

Homogeneity implies Tameness

Zhang Yingbo, Beijing Normal University

Xu Yunge, Hubei University

Introduction

Throughout the paper we always assume that k is an algebraically closed field, all the rings or algebras contain identities. We write our maps on either the left or the right, but always compose them as if they were written on the right.

In 1977 Drozd showed in [D1] that a finite-dimensional algebra over an algebraically closed field is either of tame representation type or of wild representation type, which has been one of foundations of representation theory of finite dimensional algebras.

Definition 1 [D1, CB1, DS] A finite-dimensional k -algebra Λ is of tame representation type, if for any positive integer d , there are a finite number of localizations $R_i = k[x, \phi_i(x)^{-1}]$ of $k[x]$ and Λ - R_i -bimodules T_i which are free as right R_i -modules, such that almost all (except finitely many) indecomposable Λ -modules of dimension at most d are isomorphic to

$$T_i \otimes_{R_i} R_i / (x - \lambda)^m,$$

for some $\lambda \in k$, $\phi_i(\lambda) \neq 0$, and some positive integer m .

Definition 2 [D1, CB1] A finite-dimensional k -algebra Λ is of wild representation type if there is a finitely generated Λ - $k\langle x, y \rangle$ -bi-module T , which is free as a right $k\langle x, y \rangle$ -module, such that the functor

$$T \otimes_{k\langle x, y \rangle} - : k\langle x, y \rangle\text{-mod} \rightarrow \Lambda\text{-mod}$$

preserves indecomposability and isomorphism classes.

The proof of the Drozd's Tame-Wild Theorem is highly indirect. The argument relies on the notion of a boc (the brief for "bi-module of co-algebra structure"), introduced first by Rojter in [Ro]. In 1988, Crawley-Boevey formalized the theory of bocses and showed that for a tame algebra Λ , and for each dimension d , all but finitely many isomorphism classes of indecomposable Λ -modules of dimension d are isomorphic to their Auslander-Reiten translations and hence belong to homogeneous tubes.

Since then, many authors have tried to prove the converse of Crawley-Boevey's Theorem. They expected to find infinitely many non-isomorphic indecomposable representations $\{M_i \mid i \in I\}$ of the same dimension in the representation category of a layered boc such that $M_i \not\cong DTr(M_i)$. However, four authors constructed a strongly homogeneous wild layered locs \mathfrak{B} in [BCLZ] in 2000 such that each representation of \mathfrak{B} is homogeneous (i.e., $DTr(M) \cong M$). It shows that the converse of the Crawley-Boevey Theorem does not hold true for layered bocses in general. But it is still open for finite dimensional k -algebras.

Let Λ be a finite-dimensional algebra over an algebraically closed field. We say that an indecomposable Λ -module M is *homogeneous* if $DTr(M) \cong M$. The category $\text{mod}\Lambda$ is said to be *homogeneous* provided that for each dimension d all but finitely many isomorphism classes of indecomposable Λ -modules of dimension d are homogeneous. Our purpose in this paper is to prove the following theorem

Main Theorem 3 Let Λ be a finite-dimensional algebra over an algebraically closed field. Then Λ is of tame representation type if and only if $\text{mod}\Lambda$ is homogeneous.

The necessity of the theorem is proved by Crawley-Boevey. Our proof for the sufficiency relies on the notions of matrix bi-module problem and its associated boc, as well as their reduction techniques. The key of the argument is to find a full subcategory of representation category of a bipartite matrix bi-module problem which admits infinitely many non-homogeneous representations of dimension d , and the fact that matrix bi-module problems associated to finite-dimensional algebras are bipartite.

1 Matrix Bi-module Problems

Matrix problems, which include special cases the representation theory of finite-dimensional algebras, subspace problems and projective modules, get their importance in studying questions about representation type. In this section we will introduce the notions of matrix bi-module problems over some minimal algebras and their associated bi-co-module problems, which seems to be more convenient for calculational purposes. We will unify some established notions for matrix problems, such as linear matrix problem [S], boc [Ro, D1, CB1], differential graded category [Ro] and so on, to formalize the matrix methods and approach to representation theory.

§1.1 Matrix bi-module problems

In the present subsection, we will construct a k -algebra Δ of a minimal algebra R , then define matrix bi-module problems over Δ . The concepts and the results are proposed by S. Liu.

Let \mathcal{T} be a vertex set whose elements are divided into two disjoint families: the subset \mathcal{T}_0 of trivial vertices and the subset \mathcal{T}_1 of non-trivial vertices. To each $X \in \mathcal{T}_1$, we associate an indeterminate x and a fixed non-zero polynomial $\phi_X(x)$ in $k[x]$. Given any $X \in \mathcal{T}$, we define a k -algebra R_X with identity 1_X by $R_X = k$ if X is trivial; and otherwise, $R_X = k[x, \phi_X(x)^{-1}]$ the localization of $k[x]$ at $\phi_X(x)$, and x is said to be the *parameter associated to* $X \in \mathcal{T}_1$. Now we call the k -algebra $R = \prod_{X \in \mathcal{T}} R_X$ a *minimal algebra* over \mathcal{T} with a set of orthogonal primitive idempotents $\{1_X \mid X \in \mathcal{T}\}$.

Define a tensor product of $p \geq 1$ copies of R over k :

$$R^{\otimes p} = R \otimes_k \cdots \otimes_k R = \sum_{(X_1, \dots, X_p) \in \Pi^p \mathcal{T}} R_{X_1} \otimes_k \cdots \otimes_k R_{X_p}. \quad (1.1-1)$$

There exists a natural left and right R -module structure on $R^{\otimes p}$: for any $\alpha = r_1 \otimes_k r_2 \otimes_k \cdots \otimes_k r_{p-1} \otimes_k r_p \in R^{\otimes p}$, $s \in R$, the left module action is given by: $s \otimes_R \alpha = (s \otimes_R r_1) \otimes_k r_2 \otimes_k \cdots \otimes_k r_p$; and the right one by: $\alpha \otimes_R s = r_1 \otimes_k \cdots \otimes_k r_{p-1} \otimes_k (r_p \otimes_R s)$. If $r_i \in R_{X_i}$, $s \in R_Y$, then $s \otimes_R \alpha = 0$ for $Y \neq X_1$ and $\alpha \otimes_R s = 0$ for $Y \neq X_p$. Thus $R^{\otimes p}$ can be viewed as an R - R -bi-module, or simply an $R^{\otimes 2}$ -module, with the module action:

$$(r \otimes_k s) \otimes_{R^{\otimes 2}} \alpha = r \otimes_R \alpha \otimes_R s = \alpha \otimes_{R^{\otimes 2}} (r \otimes_k s), \quad \forall r, s \in R. \quad (1.1-2)$$

Consider the direct sum of $R^{\otimes p}$, which is still an $R^{\otimes 2}$ -module:

$$\Delta = \bigoplus_{p=1}^{\infty} R^{\otimes p}, \quad \text{let } \bar{\Delta} = \bigoplus_{p=2}^{\infty} R^{\otimes p}, \quad \Delta = R \oplus \bar{\Delta}. \quad (1.1-3)$$

We define a multiplication on $R^{\otimes 2}$ -module Δ , given by $\Delta \times \Delta \rightarrow \Delta \otimes_R \Delta \subseteq \Delta$:

$$\Delta^{\otimes p} \otimes_R \Delta^{\otimes q} \subseteq \Delta^{\otimes (p+q-1)}, \quad \alpha \otimes_R \beta = r_1 \otimes_k \cdots \otimes_k (r_p s_1) \otimes_k s_2 \cdots \otimes_k s_q, \quad (1.1-4)$$

where $\beta = s_1 \otimes_k \cdots \otimes_k s_q$, and if $r_i \in R_{X_i}$, $s_j \in R_{Y_j}$, $\alpha \otimes_R \beta = 0$ for $X_p \neq Y_1$. Thus we obtain an *associative non-commutative k -algebra* $(\Delta, \otimes_R, 1_R)$ with the set of orthogonal primitive idempotents $\{1_X \mid X \in \mathcal{T}\}$.

Moreover $\Delta \otimes_R \Delta$ can be viewed as an $R^{\otimes 3}$ -module: for any $\alpha, \beta \in \Delta, r, s, w \in R$,

$$\begin{aligned} (r \otimes_k s \otimes_k w) \otimes_{R^{\otimes 3}} (\alpha \otimes_R \beta) &= (\alpha \otimes_k \beta) \otimes_{R^{\otimes 3}} (r \otimes_k s \otimes_k w) \\ &= r \otimes_R \alpha \otimes_R s \otimes_R \beta \otimes_R w. \end{aligned} \quad (1.1-5)$$

Denote by $\text{IM}_{m \times n}(\Delta)$ the set of *matrices over* Δ of size $m \times n$; and by $\mathbb{T}_n(\Delta), \mathbb{N}_n(\Delta), \mathbb{D}(\Delta)$ the set of $n \times n$ -upper triangular, strictly upper triangular, and diagonal Δ -matrices respectively. The product of two Δ -matrices is the *usual matrix product*. If $H = (h_{ij}) \in \text{IM}_{m \times n}(R)$, $U = (u_{ij}) \in \text{IM}_{m \times n}(R \otimes_k R)$, $\alpha \in \Delta$, define

$$\begin{aligned} H \otimes_R \alpha &= (h_{ij} \otimes_R \alpha) \in \text{IM}_{m \times n}(\Delta), \\ \alpha \otimes_R H &= (\alpha \otimes_R h_{ij}) \in \text{IM}_{m \times n}(\Delta); \\ U \otimes_{R^{\otimes 2}} \alpha &= (\alpha \otimes_{R^{\otimes 2}} u_{ij}) = \alpha \otimes_{R^{\otimes 2}} U \in \text{IM}_{m \times n}(\Delta). \end{aligned} \quad (1.1-6)$$

Since for any $u, v \in R^{\otimes 2}, \delta \in R^{\otimes 3}, ((\alpha \otimes_{R^{\otimes 2}} u) \otimes_R (\beta \otimes_{R^{\otimes 2}} v)) \otimes_{R^{\otimes 3}} \delta = (\alpha \otimes_R \beta) \otimes_{R^{\otimes 3}} ((u \otimes_R v) \otimes_{R^{\otimes 3}} \delta)$, if $V = (v_{jl}) \in \text{IM}_{n \times r}(R \otimes_k R)$, $\beta \in \Delta$, we have

$$(\alpha \otimes_{R^{\otimes 2}} U)(\beta \otimes_{R^{\otimes 2}} V) = (\alpha \otimes_R \beta) \otimes_{R^{\otimes 3}} (UV). \quad (1.1-7)$$

An R - R -bi-module \mathcal{S}_1 is said to be a *quasi-free bi-module* finitely generated by U_1, \dots, U_m , if the morphism

$$(R_{X_1} \otimes_k R_{Y_1}) \oplus \dots \oplus (R_{X_m} \otimes_k R_{Y_m}) \rightarrow \mathcal{S}_1, \quad 1_{X_i} \otimes_k 1_{Y_i} \mapsto U_i$$

is an isomorphism. In this case, $\{U_1, \dots, U_m\}$ is called an R - R -*quasi-free basis* of \mathcal{S}_1 .

Let R - R -bi-module $\mathcal{S}_p = R^{\otimes(p+1)} \otimes_{R^{\otimes 2}} \mathcal{S}_1$ be given by $(r \otimes_k s) \otimes_{R^{\otimes 2}} (\alpha \otimes_{R^{\otimes 2}} U) = (r \alpha s) \otimes_{R^{\otimes 2}} U = \alpha \otimes_{R^{\otimes 2}} ((r \otimes_k s) \otimes_{R^{\otimes 2}} U)$, for $r, s \in R, \alpha \in R^{p+1}, U \in \mathcal{S}$, since it is valid on the basis. Moreover, $\mathcal{S} = \sum_{p=1}^{\infty} \mathcal{S}_p = \bar{\Delta} \otimes_{R^{\otimes 2}} \mathcal{S}_1$ is an R - R -bi-module, and \mathcal{S}_p is said to be *index* p .

Definition 1.1.1 Let $T = \{1, 2, \dots, t\}$ be a set of integers, and let \sim be an equivalent relation on T , such that there is a one-to-one correspondence between the set T/\sim of equivalence classes and the set \mathcal{T} of vertices of a minimal algebra R . We may write $\mathcal{T} = T/\sim$.

Definition 1.1.2 (i) Define an R - R -bi-module $\mathcal{K}_0 = \{\text{diag}(s_{11}, \dots, s_{tt}) \mid s_{ii} \in R_X, \forall i \in X; \text{ and } s_{ii} = s_{jj}, \forall i \sim j\}$, which is isomorphic to R as algebras. Set $E_X \in \mathbb{D}_t(R_X)$ with the element $s_{ii} = 1_X$ if $i \in X$ and $s_{ii} = 0$ if $i \notin X$, then $\{E_X \mid X \in \mathcal{T}\}$ is called a quasi-free R -basis of \mathcal{K}_0 .

(ii) Define a quasi-free R - R -bi-module $\mathcal{K}_1 \subseteq \mathbb{N}(R \otimes_k R)$ with an R - R -quasi-basis:

$$\mathcal{V} = \cup_{(X,Y) \in \mathcal{T} \times \mathcal{T}} \mathcal{V}_{XY} = \{V_1, V_2, \dots, V_m\}, \quad \mathcal{V}_{XY} \subset \mathbb{N}_t(R_X \otimes_k R_Y).$$

Write $1_X V 1_Y = V$ for $V \in \mathcal{V}_{XY}$.

(iii) Suppose $\mathcal{K} = \mathcal{K}_0 \oplus (\bar{\Delta} \otimes_{R^{\otimes 2}} \mathcal{K}_1)$ possesses an algebra structure, where multiplication $m : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ is the usual matrix product over Δ ; unit $e : R \cong \mathcal{K}_0 \hookrightarrow \mathcal{K}$.

The k algebra \mathcal{K} is said to be *finitely generated in index* $(0, 1)$ over Δ , because $V_i V_j \in R^{\otimes 3} \otimes_{R^{\otimes 2}} \mathcal{K}_1$. $E_X V_j = V_j$ if $1_X V_j = V_j$, or 0 otherwise, and similarly for $V_j E_X$, thus $\{E_X \mid X \in \mathcal{T}\}$ is a set of orthogonal primitive idempotents of \mathcal{K} , $E = e(1_R) = \sum_{X \in \mathcal{T}} E_X$ is the *identity matrix*. $m_{pq} : \mathcal{K}_p \times \mathcal{K}_q \rightarrow \mathcal{K}_{p+q}$, since $R^{\otimes(p+1)} \otimes_R R^{\otimes(q+1)} \simeq R^{\otimes(p+q+1)}$.

Let $T = \{1, 2, \dots, t\}$ and $T' = \{1, 2, \dots, t'\}$ be two sets of integers. An *order* on $T \times T'$ is defined as follows: $(i, j) \preccurlyeq (i', j')$ provided that $i > i'$, or $i = i'$ but $j \leq j'$. Thus we obtain an order on the index set of entries of a matrix in $\text{IM}_{t \times t'}(\Delta)$. Let $M = (\lambda_{ij}) \in \text{IM}_{t \times t'}(\Delta)$, λ_{pq} is said to be the *leading entry* of M if $\lambda_{pq} \neq 0$, and any $\lambda_{ij} \neq 0$ implies that $(p, q) \preccurlyeq (i, j)$. Let $\bar{M} = (C_{ij})$ be a partitioned matrix over Δ , one defines similarly the *leading block* of \bar{M} . In both cases, the pair (p, q) is called the *leading position* of M resp. \bar{M} .

Let \mathcal{S} be a subspace of $\mathbb{M}_t(k)$. An ordered basis $\mathcal{U} = \{U_1, U_2, \dots, U_r\}$ with the leading positions $(p_1, q_1), \dots, (p_r, q_r)$ respectively is called a *normalized basis* of \mathcal{S} provided that

- (i) the leading entry of U_i is 1;
 - (ii) the (p_i, q_i) -entry of U_j is 0 for $j \neq i$;
 - (iii) $U_i \preceq U_j$ if and only if $(p_i, q_i) \preceq (p_j, q_j)$.
- (1.1-8)

The basis \mathcal{U} is a linear ordered set. It is easy to see that \mathcal{S} has a normalized basis by Linear algebra. In fact, taken t^2 variables x_{ij} under the order of matrix indices defined as above. Then \mathcal{S} will be the solution space of some system of linear equations

$$\sum_{(i,j) \in \mathcal{T} \times \mathcal{T}} a_{ij}^l x_{ij} = 0, \quad a_{ij}^l \in k, \quad 1 \leq l \leq s$$

for some positive integer s . Reducing the coefficient matrix to the simplest echelon form, we assume that $x_{p_1 q_1}, x_{p_2 q_2}, \dots, x_{p_r q_r}$ are all the free variables. Evaluated $x_{p_i q_i}$ the column vector whose (p_i, q_i) -entry is 1 and $(p, q) \prec (p_i, q_i)$ -entry is 0 for $i = 1, 2, \dots, r$, we obtain a normalized basis of \mathcal{S} .

Definition 1.1.3 (i) Define a quasi-free R - R -bi-module $\mathcal{M}_1 \subseteq \mathbb{M}_t(R \otimes_k R)$, such that $E_X \mathcal{M}_1 E_Y$ has a normalized basis $\mathcal{A}_{XY} \subseteq \mathbb{M}_t(k 1_X \otimes_k 1_Y k) \simeq \mathbb{M}_t(k)$, write $1_X A 1_Y = A$ for $A \in \mathcal{A}_{XY}$. Thus there is a *normalized basis*:

$$\mathcal{A} = \cup_{(X,Y) \in \mathcal{T} \times \mathcal{T}} \mathcal{A}_{XY} = \{A_1, A_2, \dots, A_n\}.$$

(ii) Let $\mathcal{M} = \bar{\Delta} \otimes_{R \otimes^2} \mathcal{M}_1$, and the algebra \mathcal{K} be given by definition 1.1.2. Define an \mathcal{K} - \mathcal{K} -bi-module structure on \mathcal{M} , with a left module action $l : \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{M}$ and a right one $r : \mathcal{M} \times \mathcal{K} \rightarrow \mathcal{M}$ given by the usual matrix product.

$l_{pq} : \mathcal{K}_p \times \mathcal{M}_q \rightarrow \mathcal{M}_{p+q}$ and $r_{pq} : \mathcal{M}_p \times \mathcal{K}_q \rightarrow \mathcal{M}_{p+q}$. The \mathcal{K} - \mathcal{K} -bi-module \mathcal{M} is said to be *finitely generated in index* $(0, 1)$ with $\mathcal{M}_0 = \{0\}$.

Definition 1.1.4 Let $H = \sum_{X \in \mathcal{T}} H_X \in \mathbb{M}_t(R)$ be a matrix, $H_X = (h_{ij})_{t \times t} \in E_X \mathbb{M}(R) E_X$ with $h_{ij} \in R_X$ for $i, j \in X$, and $h_{ij} = 0$ otherwise. Suppose H yields a derivation $d : \mathcal{K} \rightarrow \mathcal{M}$, $U \mapsto UH - HU$, such that $d(\mathcal{K}_0) = \{0\}$; $d(\mathcal{K}_1) \subseteq \mathcal{M}_1$. Clearly $d_p : \mathcal{K}_p \rightarrow \mathcal{M}_p$.

Definition 1.1.5 A quadruple $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ is called a *matrix bi-module problem* provided

- (i) R is a minimal algebra with a vertex set \mathcal{T} ;
- (ii) \mathcal{K} is an algebra given by Definition 1.1.2;
- (iii) \mathcal{M} is a \mathcal{K} - \mathcal{K} -bi-module given by Definition 1.1.3;
- (iv) There is a derivation $d : \mathcal{K} \rightarrow \mathcal{M}$ given by Definition 1.1.4.

In particular, if $\mathcal{M} = 0$, \mathfrak{A} is said to be a *minimal matrix bi-module problem*.

§1.2 Bi-co-module problems and Bocses

We will define a notion of bi-co-module problems associated to matrix bi-module problems, which is the transition into bocses. The concepts and the proofs are proposed by Y. Han.

Since \mathcal{K}_1 and \mathcal{M}_1 are both quasi-free R - R -bi-modules, we have their R - R -dual structures \mathcal{C}_1 and \mathcal{N}_1 with quasi-free R - R -quasi-basis \mathcal{V}^* and \mathcal{A}^* respectively:

$$\begin{aligned} \mathcal{C}_1 &= \text{Hom}_{R \otimes^2}(\mathcal{K}_1, R^{\otimes 2}), & \mathcal{V}^* &= \{v_1, v_2, \dots, v_m\}; \\ \mathcal{N}_1 &= \text{Hom}_{R \otimes^2}(\mathcal{M}_1, R^{\otimes 2}), & \mathcal{A}^* &= \{a_1, a_2, \dots, a_n\}. \end{aligned} \quad (1.2-1)$$

Write $v : X \mapsto Y$ (resp. $a : X \mapsto Y$) provided $1_X V 1_Y = V$ (resp. $1_X A 1_Y = A$).

Definition 1.2.1 Let \mathcal{K} be a k -algebra as in Definition 1.1.2. We define a quasi-free R -module $\mathcal{C}_0 = \sum_{X \in \mathcal{T}} R_X e_X \simeq R$ with an R -quasi-basis $\{e_X\}_{X \in \mathcal{T}}$; and a quasi-free R - R -bi-module \mathcal{C}_1 with an R - R -quasi-basis \mathcal{V}^* defined by the first formula of (1.2-1). Write $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$, define a co-algebra structure $\varepsilon : \mathcal{C} \rightarrow R$, $e_X \mapsto 1_X$, $v_j \mapsto 0$ and $\mu : \mathcal{C} \mapsto \mathcal{C} \otimes_R \mathcal{C}$ dual to $(m_{00}, m_{01}, m_{10}, m_{11})$:

$$\begin{aligned} \mu &= \begin{pmatrix} \mu_{00} \\ \mu_{10} + \mu_{01} + \mu_{11} \end{pmatrix} : \begin{pmatrix} \mathcal{C}_0 \\ \mathcal{C}_1 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{C}_0 \otimes_R \mathcal{C}_0, \\ \mathcal{C}_1 \otimes_R \mathcal{C}_0 \oplus \mathcal{C}_0 \otimes_R \mathcal{C}_1 \oplus \mathcal{C}_1 \otimes_R \mathcal{C}_1 \end{pmatrix} \\ \mu_{00}(e_X) &= e_X \otimes_R e_X; \quad \mu_{10}(v_l) = v_l \otimes_R e_{t(v_l)}; \quad \mu_{01}(v_l) = e_{s(v_l)} \otimes_R v_l; \\ \mu_{11}(v_l) &= \sum_{i,j} \gamma_{ijl} \otimes_{R^{\otimes 3}} (v_i \otimes_R v_j), \text{ if } V_i V_j = \sum_l \gamma_{ijl} \otimes_{R^{\otimes 2}} V_l. \end{aligned}$$

Definition 1.2.2 Let \mathcal{M} be a \mathcal{K} - \mathcal{K} -bi-module of Definition 1.1.2, we define a quasi free R - R -bi-module \mathcal{N}_1 with a quasi-basis \mathcal{A}^* given by the second formula of (1.2-1). Write $\mathcal{N} = \mathcal{N}_1$, suppose $V_i A_j = \sum_l \eta_{ijl} \otimes_{R^{\otimes 2}} V_l$; $A_i V_j = \sum_l \sigma_{ijl} \otimes_{R^{\otimes 2}} V_l$, then \mathcal{N} has a \mathcal{C} - \mathcal{C} -bi-co-module structure with the left and right co-module actions dual to (l_{01}, l_{11}) and (r_{10}, r_{01}) respectively:

$$\begin{aligned} \iota &= (\iota_0 + \iota_1) : \mathcal{N} \rightarrow \mathcal{C} \otimes_R \mathcal{N} = \mathcal{C}_0 \otimes_R \mathcal{N} \oplus \mathcal{C}_1 \otimes_R \mathcal{N}, \\ \iota_0(a_l) &= e_{s(a_l)} \otimes_R a_l, \quad \iota_1(a_l) = \sum_{i,j} \eta_{ijl} \otimes_{R^{\otimes 3}} (v_i \otimes_R a_j); \\ \tau &= (\tau_0 + \tau_1) : \mathcal{N} \rightarrow \mathcal{N} \otimes_R \mathcal{C} = \mathcal{N} \otimes_R \mathcal{C}_0 \oplus \mathcal{N} \otimes_R \mathcal{C}_1, \\ \tau_0(a_l) &= a_l \otimes_R e_{t(a_l)}, \quad \tau_1(a_l) = \sum_{i,j} \sigma_{ijl} \otimes_{R^{\otimes 3}} (a_i \otimes_R v_j). \end{aligned}$$

Definition 1.2.3 Let H be a matrix over R in Definition 1.1.4, and \mathcal{C}, \mathcal{N} be given in 1.2.1-1.2.2. The map $\partial = (\partial_0, \partial_1) : \mathcal{N} \rightarrow \mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ with $\partial_0 = 0$, and $\partial_1(a_l) = \sum_i \zeta_{il} \otimes_{R^{\otimes 2}} v_i$ if $d(v_i) = \sum_l \zeta_{il} \otimes_{R^{\otimes 2}} A_l$ is a co-derivation dual to (d_0, d_1) , such that $\mu\partial = (\mathbb{1} \otimes \partial)\iota + (\partial \otimes \mathbb{1})\tau$.

Definition 1.2.4 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bi-module problem. A quadruple $\mathfrak{C} = (R, \mathcal{C}, \mathcal{N}, \partial)$ is said to be a *bi-co-module problem associated to \mathfrak{A}* provided

- (i) R is a minimal algebra with a vertex set \mathcal{T} ;
- (ii) \mathcal{C} is a co-algebra given in Definition 1.2.1;
- (iii) \mathcal{N} is a \mathcal{C} - \mathcal{C} -bi-co-module given by Definition 1.2.2;
- (iv) $\partial : \mathcal{N} \rightarrow \mathcal{C}$ is a co-derivation given by Definition 1.2.3.

Now we construct the *bocs associated to a matrix bi-module problem \mathfrak{A}* based on the bi-co-module problem \mathfrak{C} associated to \mathfrak{A} presented by Roiter in [Ro].

Write $\mathcal{N}^{\otimes p} = \mathcal{N} \otimes_R \cdots \otimes_R \mathcal{N}$ with p copies of \mathcal{N} and $\mathcal{N}^{\otimes 0} = R$. Define a tensor algebra Γ of \mathcal{N} over R , whose multiplication is given by the natural isomorphisms:

$$\Gamma = \bigoplus_{p=0}^{\infty} \mathcal{N}^{\otimes p}; \quad \mathcal{N}^{\otimes p} \otimes_R \mathcal{N}^{\otimes q} \simeq \mathcal{N}^{\otimes(p+q)}$$

Let $\Xi = \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma$ be a Γ - Γ -bi-module of co-algebra structure induced by $R \hookrightarrow \Gamma$, and denoted by $(\Xi, \mu_{\Xi}, \varepsilon_{\Xi})$. Define three R - R -bi-module maps:

$$\begin{aligned} \kappa_1 : \mathcal{N} &\xrightarrow{\iota} \mathcal{C} \otimes_R \mathcal{N} \xrightarrow{\cong} R \otimes_R \mathcal{C} \otimes_R \mathcal{N} \hookrightarrow \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma, \\ \kappa_2 : \mathcal{N} &\xrightarrow{\tau} \mathcal{N} \otimes_R \mathcal{C} \xrightarrow{\cong} \mathcal{N} \otimes_R \mathcal{C} \otimes_R R \hookrightarrow \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma, \\ \kappa_3 : \mathcal{N} &\xrightarrow{\partial} \mathcal{C} \xrightarrow{\cong} R \otimes_R \mathcal{C} \otimes_R R \hookrightarrow \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma. \end{aligned}$$

Lemma 1.2.5 [CB1] $\text{Im}(\kappa_1 - \kappa_2 + \kappa_3)$ is a Γ -co-ideal in Ξ . Thus $\Omega := \Xi / \text{Im}(\kappa_1 - \kappa_2 + \kappa_3)$ is a Γ - Γ -bi-module of co-algebra structure.

Proof Recall the law of bi-co-module: $(\mu \otimes \mathbb{1})\iota = (\mathbb{1} \otimes \iota)\iota$, $(\mathbb{1} \otimes \mu)\tau = (\tau \otimes \mathbb{1})\tau$, $(\mathbb{1} \otimes \tau)\iota = (\iota \otimes \mathbb{1})\tau$, $(\mathbb{1} \otimes \partial)\iota - \mu\partial + (\partial \otimes \mathbb{1})\tau = 0$. Thus, for any $b \in \mathcal{N}$, we have

$$\mu_{\Xi}(\kappa_1 - \kappa_2 + \kappa_3)(b)$$

$$\begin{aligned}
&= \mu_{\Xi}(1_{\Gamma} \otimes \iota(b)) - \mu_{\Xi}(\tau(b) \otimes 1_{\Gamma}) + \mu_{\Xi}(\mathbb{1} \otimes \partial(b) \otimes 1_{\Gamma}) \\
&= (\mu \otimes \mathbb{1})\iota(b) - (\mathbb{1} \otimes \mu)\tau(b) + \mu(\partial(b)) \\
&= (\mathbb{1} \otimes \iota)\iota(b) - (\tau \otimes \mathbb{1})\tau(b) + (\mathbb{1} \otimes \partial)\iota(b) + (\partial \otimes \mathbb{1})\tau(b) \\
&= (\mathbb{1} \otimes \iota - \mathbb{1} \otimes \tau + \mathbb{1} \otimes \partial)\iota(b) + (\iota \otimes \mathbb{1} - \tau \otimes \mathbb{1} + \partial \otimes \mathbb{1})\tau(b) \\
&= u_{(1)} \otimes (\iota - \tau + \partial)(b_{(1)}) + (\iota - \tau + \partial)(b_{(2)}) \otimes u_{(2)} \\
&= u_{(1)} \otimes (\kappa_1 - \kappa_2 + \kappa_3)(b_{(1)}) + (\kappa_1 - \kappa_2 + \kappa_3)(b_{(2)}) \otimes u_{(2)} \\
&\in \Xi \otimes \text{Im}(\kappa_1 - \kappa_2 + \kappa_3) + \text{Im}(\kappa_1 - \kappa_2 + \kappa_3) \otimes \Xi
\end{aligned}$$

where $\iota(b) := u_{(1)} \otimes b_{(1)}$, $\tau(b) := b_{(2)} \otimes u_{(2)}$, and each term in each step is viewed as an element in $\Xi \otimes_{\Gamma} \Xi$ naturally. The proof is completed.

Recall from [CB1, 3.4 Definition], that $\mathfrak{B} = (\Gamma, \Omega)$ defined as above is a boc s with a layer

$$L = (R; \omega; a_1, a_2, \dots, a_n; v_1, v_2, \dots, v_m).$$

Denote by ε_{Ω} and μ_{Ω} the induced co-unit and co-multiplication, then $\bar{\Omega} = \ker \varepsilon_{\Omega}$ is a Γ - Γ -bi-module freely generated by v_1, v_2, \dots, v_m , and $\Omega = \Gamma \oplus \bar{\Omega}$ as bi-modules.

From this, we use the imbedding: $\mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \mathcal{N} \otimes \mathcal{C}_1 \oplus \mathcal{C}_1 \otimes \mathcal{N} \hookrightarrow \Gamma \otimes_R (\mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \mathcal{N} \otimes_R \mathcal{C}_1 \oplus \mathcal{C}_1 \otimes_R \mathcal{N}) \otimes_R \Gamma \subset \Omega$; and the isomorphism: $\bar{\Omega} \otimes_R \bar{\Omega} \simeq \bar{\Omega} \otimes_{\Gamma} \bar{\Omega}$. The group-like $\omega : R \rightarrow \Omega$, $1_X \mapsto e_X$ is an R - R -bi-module map. Since $\delta_1(a_i) = \tau_0(a_i) - \iota_0(a_i)$, and $(\iota_0(a_i) + \iota_1(a_i)) - (\tau_0(a_i) + \tau_1(a_i)) + \partial_1(a_i) = (\iota - \tau + \partial)(a_i) = (\kappa_1 - \kappa_2 + \kappa_3)(a_i) = 0$ in Ω , the pair of the differentials determined by ω is given by

$$\begin{aligned}
\delta_1 : \Gamma &\rightarrow \bar{\Omega}, 1_X \mapsto 0, a_i \mapsto \iota_1(a_i) - \tau_1(a_i) + \partial_1(a_i), \quad \forall a_i \in \mathcal{A}^*(X, Y); \\
\delta_2 : \bar{\Omega} &\mapsto \bar{\Omega} \otimes_{\Gamma} \bar{\Omega}, v_j \mapsto \mu_{11}(v_j), \quad \forall v_j \in \mathcal{V}^*(X, Y).
\end{aligned} \tag{1.2-2}$$

Recall from [CB1] that a representation of a layered boc s \mathfrak{B} is a left Γ -module P of finite dimension \underline{d} , which means that P consists of a set of vector spaces and a set of linear maps:

$$\begin{aligned}
&\{P_X = k^{d_X}, P(x) : P_X \mapsto P_X \mid X \in \mathcal{T}\}; \\
&\{P(a_i) : k^{m_{X_i}} \rightarrow k^{m_{Y_i}} \mid a_i : X_i \mapsto Y_i, 1 \leq i \leq n\}.
\end{aligned} \tag{1.2-3}$$

A morphism from P to Q is given by a Γ -map $f : \Omega \otimes_{\Gamma} P \mapsto Q$, which is equivalent to write $f = (f_X; f(v_j))_{X \in \mathcal{T}; j=1, \dots, m}$, such that

$$\begin{aligned}
P(a_l)f_{Y_l} - f_{X_l}Q(a_l) &= \sum_{i,j} \eta_{ijl} \otimes_{R^{\otimes 3}} (f(v_i) \otimes_R Q(a_j)) \\
&\quad - \sum_{i,j} \sigma_{ijl} \otimes_{R^{\otimes 3}} (P(a_i) \otimes_R f(v_j)) + \sum_i \zeta_{il} \otimes_{R^{\otimes 2}} f(v_i)
\end{aligned} \tag{1.2-4}$$

for all $a_l \in \mathcal{A}^*$, $1 \leq l \leq n$, by substituting $P(x)$ for x , see [BK]. It is clear that

$$\text{Hom}_{\Gamma}(\bar{\Omega} \otimes_{\Gamma} P, Q) \simeq \oplus_{j=1}^m \text{Hom}_{\Gamma}(\Gamma 1_{s(v_j)} \otimes_k 1_{t(v_j)} P, Q) \simeq \oplus_{j=1}^m \text{Hom}_k(1_{t(v_j)} P, 1_{s(v_j)} Q). \tag{1.2-5}$$

Denote by $R(\mathfrak{B})$ the representation category of \mathfrak{B} .

§1.3 The representation category of a matrix bi-module problem

This sub-section is devoted to defining the representation category of a matrix bi-module problem. Which is relatively complicated, but extremely useful for the proof of the main theorem.

Definition 1.3.1 Let $J(\lambda) = J_d(\lambda)^{e_d} \oplus J_{d-1}(\lambda)^{e_{d-1}} \oplus \dots \oplus J_1(\lambda)^{e_1}$, with e_i non-negative integers, be a Jordan matrix. Set

$$m_j = e_d + e_{d-1} + \dots + e_j.$$

The following partitioned matrix $W(\lambda)$ similar to $J(\lambda)$ is called a *Weyr matrix of eigenvalue λ* :

$$W(\lambda) = \begin{pmatrix} \lambda I_{m_1} & W_{12} & 0 & \cdots & 0 & 0 \\ & \lambda I_{m_2} & W_{23} & \cdots & 0 & 0 \\ & & \lambda I_{m_3} & \cdots & 0 & 0 \\ & & & \ddots & \vdots & \vdots \\ & & & & \lambda I_{m_{d-1}} & W_{d-1,d} \\ & & & & & \lambda I_{m_d} \end{pmatrix}_{d \times d}, \quad W_{j,j+1} = \begin{pmatrix} I_{m_{j+1}} \\ 0 \end{pmatrix}_{m_j \times m_{j+1}}.$$

A direct sum $W = W(\lambda_1) \oplus W(\lambda_2) \oplus \cdots \oplus W(\lambda_s)$ with distinct eigenvalues λ_i is said to be a *Weyr matrix*. We may define an order $<$ on the base field k , so that each Weyr matrix has a unique form. Similarly, let Z_{ij} be a set of vertices, $S = k1_{Z_{ij}} \oplus k[z, \phi(z)^{-1}]$ be a minimal algebra, $\bar{W} \simeq \oplus J_{ij}(\lambda_i)^{e_{ij}} 1_{Z_{ij}} \oplus (z)^\delta$ with $\delta = 0$ or 1 is said to be a *Weyr matrix over S* . If $S = \oplus_j k1_{Z_j}$ trivial, then $\bar{W} = \oplus_j I_{n_j} 1_{Z_j}$ is still called a Weyr matrix over S .

Form now on, we assume that $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ is a matrix bi-module problem with $T = \{1, 2, \dots, t\}$ and its partition \mathcal{T} . Let $X \in \mathcal{T}_1$. A Weyr matrix W over k is called R_X -regular if its eigenvalues are R_X -regular, i.e. $\phi_X(\lambda) \neq 0$, for all the eigenvalues λ . If $X \in \mathcal{T}_0$, an identity matrix I is called R_X -regular.

A vector of non-negative integers is called a *size vector* over T , if $\underline{m} = (m_1, m_2, \dots, m_t)$ with $m_i = m_j, \forall i \sim j$. $m = \sum_{i=1}^t m_i$ is said to be the *size* of \underline{m} . Given a size vector \underline{m} , the vector $\underline{d} = (m_X)_{X \in \mathcal{T}}$ with $m_X = m_i, \forall i \in X$ is called a dimension vector *over \mathcal{T}* determined by \underline{m} .

Definition 1.3.2 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bi-module problem, let S be a minimal algebra, $\Sigma = \oplus_{p=1}^\infty S^{\otimes p}$, see Formula (1.1-3).

(i) Write $H_X = (h_{ij}(x)1_X)_{t \times t}$ with $h_{ij}(x) \in k[x]$ for $X \in \mathcal{T}_1$, and $x = 1$ for $X \in \mathcal{T}_0$. Let \bar{W}_X be a Weyr matrix of size m_X over S . We define an $\underline{m} \times \underline{m}$ -partitioned matrix:

$$H_X(\bar{W}_X) = (B_{ij})_{t \times t}, \quad B_{ij} = \begin{cases} h_{ij}(\bar{W}_X)_{m_i \times m_j}, & i, j \in X, \\ (0)_{m_i \times m_j}, & i \notin X \text{ or } j \notin X, \end{cases}$$

(ii) Let $\underline{m} = (m_1, \dots, m_t)$ and $\underline{n} = (n_1, \dots, n_t)$ be two size vectors over \mathcal{T} , and let $F \in \text{IM}_{m_X \times n_X}(S^{\otimes p}), p = 1, 2$, with an R_X - R_X -module structure. The star product $*$ of F_X and E_X is defined to be a diagonal $\underline{m} \times \underline{n}$ -partitioned matrix:

$$F_X * E_X = (B_{ij})_{t \times t}, \quad B_{ii} = \begin{cases} F_X, & i \in X, \\ 0 & i \notin X, \end{cases}$$

(iii) Let $U = V_j$ or A_i for some $j = 1, \dots, m, i = 1, \dots, n$, and $U = (\alpha_{ij}) \in E_X \text{IM}_t(R \otimes_k R)E_Y$; let $C \in \text{IM}_{m_X \times n_Y}(S^{\otimes p})$ with $p = 2$, and possibly $p = 1$ for $U = A_1$; $\{\bar{W}_X \in \text{IM}_{m_X}(S) \mid X \in \mathcal{T}\}$, $\{W'_Y \in \text{IM}_{n_Y}(S) \mid Y \in \mathcal{T}\}$ be two sets of regular Weyr matrices. Suppose there is an R_X - R_Y -bi-module structure on $\text{IM}_{m_X \times n_Y}(S^{\otimes p})$:

$$(x \otimes_k y) \otimes_{R^{\otimes 2}} C = W_X C W'_Y. \quad (1.3-1)$$

The star product $*$ of C and U is defined to be a $(\underline{m} \times \underline{n})$ -partitioned matrix:

$$C * U = (B_{ij})_{t \times t}, \quad B_{ij} = \begin{cases} C \otimes_{R^{\otimes 2}} \alpha_{ij}, & i \in X, j \in Y; \\ 0_{m_i \times n_j}, & i \notin X, \text{ or } j \notin Y. \end{cases}$$

Lemma 1.3.3 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bi-module problem.

(i) If $C \in \text{IM}_{m_X \times n_Y}(S^{\otimes 2})$, $\zeta_{il} \in R_X \otimes_k R_Y$ are given in Definition 1.2.3, then by the usual product of Σ -matrices:

$$(C * V_i)H_Y(\bar{W}_Y) - H_X(\bar{W}_X)(C * V_i) = \sum_l (C \otimes_{R^{\otimes 2}} \zeta_{il}) * A_l.$$

(ii) If $F_X \in \text{IM}_{m_X \times n_X}(S^{\otimes p})$, $p = 1, 2$; $C \in \text{IM}_{m_X \times n_Y}(S^{\otimes q})$, where $q = 2$, and possibly $q = 1$ for $U = A_1$, then by the usual multiplication of matrices over Σ :

$$(F_X * E_X)(C * U) = \begin{cases} (F_X C) * U, & 1_X U = U; \\ 0, & 1_X U = 0. \end{cases}$$

Similarly, $(C * U)(F_X * E_X) = (C F_X) * U$ if $U 1_X = U$, or 0 if $U 1_X = 0$.

(iii) Let $U \in E_X \text{IM}_t(R \otimes_k R) E_Y$, $V \in E_Y \text{IM}_t(R \otimes_k R) E_Z$ and $UV = \sum_{l=1}^n \epsilon_l \otimes_{R^{\otimes 2}} G_l$ with $\epsilon_l \in R_X \otimes_k R_Y \otimes_k R_Z$ and $G_l \in E_X \text{IM}_t(R \otimes_k R) E_Z$. Let $\underline{m}, \underline{n}, l$ be size vectors over \mathcal{T} , the $R \otimes_k R$ -module structures given by Formula (1.3-1) yield an $R^{\otimes 3}$ -module structure on $\oplus_{(X,Y,Z) \in \mathcal{T} \times \mathcal{T}} \text{IM}_{m_X n_Y}(S^{\otimes p}) \otimes_R \text{IM}_{n_Y l_Z}(S^{\otimes q})$ for $p, q = 2$, and possibly $p = 1$ for $U = A_1$, or $q = 1$ for $V = A_1$. Then by the usual Σ -matrix product:

$$(C * U)(D * V) = \sum_{l=1}^n ((C \otimes_R D) \otimes_{R^{\otimes 3}} \epsilon_l) * G_l.$$

Proof (i) Write $H_X = (\gamma_{pq})$, $\gamma \in R_X$; $H_Y = (\delta_{pq})$, $\delta \in R_Y$; $V_i = (\alpha_{pq})$, $\alpha \in R_X \otimes_k R_Y$. The left side = $(\sum_l (C \otimes_{R^{\otimes 2}} \alpha_{pl}) \otimes_R \delta_{lq}) - (\sum_l \gamma_{pl} \otimes_R (C \otimes_{R^{\otimes 2}} \alpha_{lq})) = (C \otimes_{R^{\otimes 2}} \sum_l (\alpha_{pl} \delta_{lq} - \gamma_{pl} \alpha_{lq})) = C * (V_i H_Y - H_X V_i) = C * d(V_i) = C * (\sum_l \zeta_{il} A_l) = \text{the right side.}$

(ii) Write $U = (\alpha_{pq})$, $\alpha \in R \otimes_k R$, the left side = $(F_X 1_X (C \otimes_{R^{\otimes 2}} \alpha_{pq})) = ((F_X C) \otimes_{R^{\otimes 2}} \alpha_{pq}) = \text{the right side.}$

(iii) Write $U = (\alpha_{pq})$, $\alpha \in R_X \otimes_k R_Y$, $V = (\beta_{pq})$, $\beta \in R_Y \otimes_k R_Z$. The left side = $(\sum_l (C \otimes_{R^{\otimes 2}} \alpha_{pl}) (D \otimes_{R^{\otimes 2}} \beta_{lq})) = ((C D) \otimes_{R^{\otimes 3}} (\sum_l \alpha_{pl} \otimes_R \beta_{lq})) = \text{the right side.}$ The proof is finished.

Definition 1.3.4 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bi-module problem, and \underline{m} a size vector over \mathcal{T} . Then a representation \bar{P} of \mathfrak{A} can be written as an $\underline{m} \times \underline{m}$ -partitioned matrix over k :

$$\bar{P} = \sum_{X \in \mathcal{T}} H_X(W_X) + \sum_{i=1}^n \bar{P}(a_i) * A_i,$$

where $W_X \in \text{IM}_{m_X}(k)$ is regular for any $X \in \mathcal{T}$, $\bar{P}(a_i) \in \text{IM}_{m_{X_i} \times m_{Y_i}}(k)$ see Formula (1.2-3). Taken $S = k = \Sigma$, the first summand is defined in 1.3.2 (i), and the second one in (iii).

Definition 1.3.5 Let \underline{m}, \bar{P} be given above, \underline{n} be a size vector over \mathcal{T} , and \bar{Q} a representation over \mathfrak{A} . A morphism $\bar{f} : \bar{P} \rightarrow \bar{Q}$ can be written as an $\underline{m} \times \underline{n}$ -partitioned matrix by Definition 1.3.2 (ii) and (iii) for $S = k = \Sigma$:

$$\bar{f} = \sum_{X \in \mathcal{T}} \bar{f}_X * E_X + \sum_{j=1}^m \bar{f}(v_j) * V_j,$$

where $\bar{f}_X \in \text{IM}_{m_X \times n_X}(k)$, $\bar{f}(v_j) \in \text{IM}_{m_{s(v_j)} \times n_{t(v_j)}}(k)$, such that $\bar{P} \bar{f} = \bar{f} \bar{Q}$, where the multiplication is given according to Lemma 1.3.3 (i)-(iii).

If $\bar{f}' : \bar{Q} \rightarrow \bar{U}$ is also a morphisms over \mathfrak{A} . Then $\bar{f} \bar{f}' : \bar{P} \rightarrow \bar{U}$ calculated according to Lemma 1.3.3 (ii)-(iii) is a morphism. In fact, $(\bar{f} \bar{f}') \bar{P} = \bar{f}(\bar{Q} \bar{f}') = (\bar{U} \bar{f}) \bar{f}' = \bar{U}(\bar{f} \bar{f}')$. We denote by $R(\mathfrak{A})$ the category of representations of the matrix bi-module problem \mathfrak{A} .

Theorem 1.3.6 Let \mathfrak{A} be a matrix bi-module problem, and \mathfrak{B} the associated boc. Then the categories $R(\mathfrak{A})$ and $R(\mathfrak{B})$ are equivalent.

Proof Without loss of generality, we may assume that $\{P(x) = W_X \mid X \in \mathcal{T}\}$ is a set of regular Weyr matrices. Then \bar{P} in Definition 1.3.4 and P in Formula (1.2-3) are one-to-on correspondent; \bar{f} of Definition 1.3.5 and f in Formula (1.2-5) are one-to-on correspondent. Moreover, $\bar{P} \bar{f} = \bar{f} \bar{Q}$ if and only if f satisfying Formula (1.2-4), for the proof of this assertion, we refer to Formula (1.4-2) and Theorem 1.4.2 below.

Thanks to Theorem 1.3.6, we will denote by P, f in both $R(\mathfrak{A})$ and $R(\mathfrak{B})$ in a unified manner.

1.4. Formal Products and Formal Equations

Now we introduce a notion of “formal equation”, which will build a nice connection between matrix bi-module problems and associated bocses.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bi-module problem, with an associated bi-co-module problem $\mathfrak{C} = (R, \mathcal{C}, \mathcal{N}, \partial)$. Recall that $\{E_X\}$ and $\{e_X\}$ are dual bases of $(\mathcal{K}_0, \mathcal{C}_0)$; $\{V_1, \dots, V_m\}$ and $\{v_1, \dots, v_m\}$ are those of $(\mathcal{K}_1, \mathcal{C}_1)$; and $\{A_1, \dots, A_n\}$ and $\{a_1, \dots, a_n\}$ of $(\mathcal{M}_1, \mathcal{N}_1)$. Then, taken $S = R, \Sigma = \Delta, \underline{m} = (1, \dots, 1) = \underline{n}$, in Definition 1.3.2 (ii)-(iii):

$$\begin{aligned}\Upsilon &= \sum_{X \in \mathcal{T}} e_X * E_X \\ \Pi &= \sum_{j=1}^m v_j * V_j \\ \Theta &= \sum_{i=1}^n a_i * A_i\end{aligned}\tag{1.4-1}$$

are called the *formal products* of $(\mathcal{K}_0, \mathcal{C}_0)$, $(\mathcal{K}_1, \mathcal{C}_1)$ and $(\mathcal{M}_1, \mathcal{N}_1)$ respectively.

Lemma 1.4.1 With the notations above, and δ the differential in the associated bocs \mathfrak{B} . Then the matrix multiplication yields:

$$\begin{aligned}(\sum_{i=1}^m v_i * V_i)(\sum_{j=1}^m v_j * V_j) &= \sum_{l=1}^m \mu_{11}(v_l) * V_l; \\ (\sum_{i=1}^n a_i * A_i)(\sum_{j=1}^m v_j * V_j) &= \sum_{l=1}^n \tau_1(a_l) * A_l; \\ (\sum_{j=1}^m v_j * V_j)(\sum_{i=1}^n a_i * A_i) &= \sum_{l=1}^n \iota_1(a_l) * A_l; \\ (\sum_{j=1}^m v_j * V_j)H - H(\sum_{j=1}^m v_j * V_j) &= \sum_{l=1}^n \partial_1(a_l) * A_l; \\ (\sum_{l=1}^n a_l * A_l)(\sum_{X \in \mathcal{T}} e_X * E_X) - (\sum_{X \in \mathcal{T}} e_X * E_X)(\sum_{l=1}^n a_l * A_l) &= \sum_{l=1}^n \delta(a_l) * A_l.\end{aligned}$$

Proof ① We first prove the second equality, the proofs of the first and the third are similar. By Lemma 1.3.3 (iii) for $S = R, p = q = 2$, the left side $= \sum_{l=1}^n (\sum_{i,j} \sigma_{ijl} \otimes_{R \otimes 3} (a_i \otimes_R v_i)) * A_l =$ the right side. ② For the fourth equality, by Lemma 1.3.3 (i) the left side $= \sum_{l=1}^n (\sum_{j=1}^m \zeta_{lj} \otimes_{R \otimes 2} v_j) * A_l =$ the right side. ③ For the last one, by Lemma 1.3.3 (ii), $p = 1, q = 2$, the left side $= \sum_{l=1}^n (a_l \otimes_R e_{Y_l} - e_{X_l} \otimes_R a_l) * A_l =$ the right side. The proof of the lemma is completed.

Denote by $(\mathfrak{A}, \mathfrak{B})$ the pair of a matrix bi-module problem and its associated bocs. Then the matrix equation $(\Theta + H)(\Upsilon + \Pi) = (\Upsilon + \Pi)(\Theta + H)$, more precisely,

$$\begin{aligned}& (\sum_{i=1}^n a_i * A_i + H)(\sum_{X \in \mathcal{T}} e_X * E_X + \sum_{j=1}^m v_j * V_j) \\ &= (\sum_{X \in \mathcal{T}} e_X * E_X + \sum_{j=1}^m v_j * V_j)(\sum_{i=1}^n a_i * A_i + H)\end{aligned}\tag{1.4-2}$$

is called the *formal equation* of $(\mathfrak{A}, \mathfrak{B})$ due to the following theorem.

Theorem 1.4.2 The entry at the leading position of A_l in the formal equation is

$$\delta(a_l) = \iota_1(a_l) - \tau_1(a_l) + \partial_1(a_l).$$

Proof. According to Formula (1.4-2) and Lemma 1.4.1:

$$\begin{aligned}& \sum_{l=1}^n \delta(a_l) * A_l = \sum_{l=1}^n (a_l e_{t(A_l)} - e_{s(A_l)} a_l) * A_l \\ &= \sum_{j,i} (v_j * V_j)(a_i * A_i) - \sum_{i,j} (a_i * A_i)(v_j * V_j) + \sum_j ((v_j * V_j)H - H(v_j * V_j)) \\ &= \sum_{l=1}^n \iota_1(a_l) * A_l - \sum_{l=1}^n \tau_1(a_l) * A_l + \sum_{l=1}^n \partial_1(a_l) * A_l \\ &= \sum_{l=1}^n (\iota_1(a_l) - \tau_1(a_l) + \partial_1(a_l)) * A_l.\end{aligned}$$

We obtain the expression at the leading position of A_l for $1 \leq l \leq n$. The proof is finished.

Moreover, the first formula of Lemma 1.4.1 gives:

$$\begin{aligned}& (\sum_{X \in \mathcal{T}} e_X * E_X + \sum_{i=1}^m v_i * V_i)(\sum_{X \in \mathcal{T}} e_X * E_X + \sum_{j=1}^m v_j * V_j) \\ &= \sum_{X \in \mathcal{T}} (e_X \otimes_R e_X) * E_X + \sum_{l=1}^m \mu(v_l) * V_l.\end{aligned}\tag{1.4-3}$$

Now we define a special class of matrix bi-module problems to end the sub-section.

Definition 1.4.3 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a matrix bi-module problem with R trivial. \mathfrak{A} is said to be *bipartite* if $\mathcal{T} = \mathcal{T}' \dot{\cup} \mathcal{T}''$, $R = R' \times R''$, $\mathcal{K} = \mathcal{K}' \times \mathcal{K}''$ as direct products of algebras, and \mathcal{M} is a \mathcal{K}' - \mathcal{K}'' -bi-module.

Let Λ be a finite-dimensional basic k -algebra, $J = \text{rad}(\Lambda)$ be the Jacobson radical of Λ with the nilpotent index m , and $S = \Lambda/J$. Suppose $\{e_1, \dots, e_h\}$ is a complete set of orthogonal primitive idempotents of Λ . Taken the pre-images of k -bases of $e_i(J^i/J^{i+1})e_j$ under the canonical projections $J^i \rightarrow J^i/J^{i+1}$ in turn for $i = m, \dots, 1$, we obtain an ordered basis of J under the length order, see [CB1, 6.1] for details. Then we construct the left regular representation $\bar{\Lambda}$ of Λ under the k -basis $(a_n, \dots, a_2, a_1, e_1, \dots, e_h)$ of Λ , which yields a bipartite matrix bi-module problem $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ with

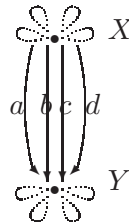
$$R = S \times S; \quad \mathcal{K}_0 \oplus \mathcal{K}_1 = \bar{\Lambda} \times \bar{\Lambda}; \quad \mathcal{M}_1 = \text{rad}(\bar{\Lambda}); \quad H = 0.$$

Remark 1.4.4 A simple calculation shows that the *row indices* of the leading positions of the base matrices in \mathcal{A} are *pairwise different*, and the *column index* of the leading position of $A \in \mathcal{A}_{XZ}$ equals $j_Z = \max\{j \in Z\}$ for any $X \in \mathcal{T}$, they are *concentrated*, the j_Z -th column is said to be a *main column* over Z . Such a fact is denoted by RDCC for short, which is not essential in the proof of the main theorem, but makes it easier and more intuitive.

Example 1.4.5 [D1, R1] Let $Q = a \circ \bullet \bullet \circ b$ be a quiver, $I = \langle a^2, ba - ab, ab^2, b^3 \rangle$ be an ideal of kQ , and $\Lambda = kQ/I$. Denote the residue classes of e, a, b in Λ still by e, a, b respectively. Moreover set $c = b^2, d = ab$. Then an ordered k -basis $\{d, c, b, a, e\}$ of Λ yields a regular representation $\bar{\Lambda}$. A matrix bi-module problem \mathfrak{A} follows by Theorem 1.4.4, with its associated boc \mathfrak{B} . The formal equation of the pair $(\mathfrak{A}, \mathfrak{B})$ can be written as:

$$\begin{pmatrix} e & 0 & u_1 & u_2 & u_4 \\ & e & u_2 & 0 & u_3 \\ & & e & 0 & u_2 \\ & & & e & u_1 \\ & & & & e \end{pmatrix} \begin{pmatrix} 0 & 0 & a & b & d \\ & 0 & b & 0 & c \\ & & 0 & 0 & b \\ & & & 0 & a \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a & b & d \\ & 0 & b & 0 & c \\ & & 0 & 0 & b \\ & & & 0 & a \\ & & & & 0 \end{pmatrix} \begin{pmatrix} f & 0 & v_1 & v_2 & v_4 \\ & f & v_2 & 0 & v_3 \\ & & f & 0 & v_2 \\ & & & f & v_1 \\ & & & & f \end{pmatrix}$$

where $e = e_X, f = e_Y$ for simplicity. Denote by A, B, C, D the R - R -quasi-basis of \mathcal{M}_1 , and by a, b, c, d the R - R -dual basis of \mathcal{N}_1 . From this we can obtain the associated boc \mathfrak{B} with the layer $L = (R; \omega; a, b, c, d; u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4)$.



$$\begin{cases} \delta(a) = 0, \\ \delta(b) = 0, \\ \delta(c) = u_2b - bv_2, \\ \delta(d) = u_1b + u_2a - bv_1 - av_2. \end{cases}$$

2 Reductions for Matrix Bi-module Problems

In the present section, we will define six reductions in terms of matrix bi-module problems associated to those of bocses, and will give two additional ones. Then we discuss defining systems of pairs, in order to construct the induced pairs in series of reductions.

§2.1 Triangular properties and admissible bi-modules

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, d)$ be a matrix bi-module problem. Then for any $A_i \in \mathcal{M}_1 \subseteq \mathbb{M}_t(R \otimes_k R)$, the leading position of $V_j A_i$ (resp. $A_i V_j$) is strictly larger than that of A_i , since $\mathcal{K}_1 \subseteq \mathbb{N}(R \otimes_k R)$. But $\mathcal{A} = \{A_1, \dots, A_n\}$ is an ordered set of normalized basis, the left and right module action satisfies Formula (2.1-1) below, so called *triangular* property:

$$l(\mathcal{K}_1 \times A_i), r(A_i \times \mathcal{K}_1) \subseteq \oplus_{l=i+1}^n R^{\otimes 3} \otimes_{R^{\otimes 2}} A_l. \quad (2.1-1)$$

Since $(\mathcal{K}_1, \mathcal{C}_1)$ and $(\mathcal{M}_1, \mathcal{N}_1)$ are dual R - R -bi-modules in the associated bi-co-module problem $\mathfrak{C} = (R, \mathcal{C}, \mathcal{N}, \partial)$ of \mathfrak{A} , the left and right co-module action also possesses the *triangular* property:

$$\iota_1(a_l) \in \oplus_{i=1}^{l-1} \mathcal{C}_1 \otimes_R a_i, \quad \tau_1(a_l) \in \oplus_{i=1}^{l-1} a_i \otimes_R \mathcal{C}_1. \quad (2.1-2)$$

Define a \mathcal{K} - \mathcal{K} sub-bi-module and the corresponding \mathcal{K} - \mathcal{K} -quotient-bi-module:

$$\mathcal{M}^{(h)} = \oplus_{i=h+1}^n \bar{\Delta} \otimes_{R^{\otimes 2}} A_i \subseteq \mathcal{M}, \quad \mathcal{M}^{[h]} = \mathcal{M} / \mathcal{M}^{(h)}.$$

$\mathfrak{A}^{[h]} = (R, \mathcal{K}, \mathcal{M}^{[h]}, \bar{d})$, with \bar{d} induced from d , is said to be a *quotient* of \mathfrak{A} , which itself might be no longer a matrix bi-module problem. If $\Gamma^{(h)}$ is freely generated by a_1, \dots, a_h , the associated bocs $\mathfrak{B} = (\Gamma, \Omega)$ has a sub-bocs:

$$\mathfrak{B}^{(h)} = (\Gamma^{(h)}, \Gamma^{(h)} \otimes_R \Omega \otimes_R \Gamma^{(h)}).$$

Note a simple fact: let $(\mathfrak{A}, \mathfrak{C}, \mathfrak{B})$ be a triple defined above, then

$$\begin{aligned} & l(\mathcal{K}_1 \times \mathcal{M}_1), r(\mathcal{M}_1 \times \mathcal{K}_1), d(\mathcal{K}_1) \subseteq \mathcal{M}_1^{(h)} \text{ in } \mathfrak{A} \\ \iff & \mathcal{C}_1 \otimes_R \mathcal{N}_1^{(h)} = 0, \mathcal{N}_1^{(h)} \otimes_R \mathcal{C}_1 = 0, \partial(\mathcal{N}_1^{(h)}) = 0 \text{ in } \mathfrak{C} \\ \iff & \delta(\Gamma^{(h)}) = 0 \text{ in } \mathfrak{B}. \end{aligned}$$

In fact, the condition in \mathfrak{A} is equivalent to $\eta_{jil} = 0, \sigma_{ijl} = 0, \zeta_{jl} = 0$ for $l = 1, \dots, h$ and any i, j , which is equivalent to the condition on \mathfrak{C} and \mathfrak{B} .

Let $R_X = k[x, \phi(x)^{-1}]$, r a fixed positive integer, and $\lambda_1, \dots, \lambda_s \in k$ with $\phi(\lambda_i) \neq 0$, write $g(x) = (x - \lambda_1) \cdots (x - \lambda_s)$. Define a minimal algebra S and a R_X -module K over S :

$$\begin{aligned} S &= (\prod_{i=1}^s \prod_{j=1}^r k 1_{Z_{ij}}) \times k[z, \phi(z)^{-1} g(z)^{-1}], \\ K &= (\oplus_{i=1}^s \oplus_{j=1}^r \oplus_{q=1}^j k 1_{Z_{ijq}}) \oplus k[z, \phi(z)^{-1} g(z)^{-1}], \\ K(x) = \bar{W} : K &\rightarrow K, \quad \bar{W} \simeq \oplus_{i=1}^s \oplus_{j=1}^r J_j(\lambda_j) 1_{Z_{ij}} \oplus (z), \end{aligned} \quad (2.1-3)$$

a Weyr matrix over S . Set $n_X = \frac{1}{2}sr(r+1) + 1$, denote by $\{(i, j, l) \mid 1 \leq l \leq j, 1 \leq j \leq r, 1 \leq i \leq s\} \cup \{n_X\}$, the index set of the direct summands of K . The order on the set is defined by

$$(i, j, l) \prec (i', j', l') \iff i < i'; \text{ or } i = i', l < l'; \text{ or } i = i', l = l', j > j'.$$

there is a partition on the index set, such that $Z_{ij} = \{(i, j, l) \mid l = 1, \dots, j\}$, $Z = \{n_X\}$. Suppose $\mathbf{e}_{(i,j,l)}$ and $\mathbf{f}_{(i,j,l)}$ are $1 \times n_X$ and $n_X \times 1$ matrices respectively, with $1_{Z_{ij}}$ at the (i, j, l) -th component and 0 at others. Let m_X square matrix $\mathbf{f} \otimes_S \mathbf{e}$ (resp. $\mathbf{f} \otimes_k \mathbf{e}$) stand for the usual matrix tensor product over S (resp. k). Let $\text{End}_{R_X}(K)$ be the endomorphism ring of K over $\Sigma = \sum_{p=1}^{\infty} S^{\otimes p}$. Then the S -quasi basis of index 0, and S - S -quasi-basis of index 1 are given respectively by

$$\{F_{ij} = \sum_{l=1}^j \mathbf{f}_{(i,j,l)} \otimes_S \mathbf{e}_{(i,j,l)}; F_{m_X m_X} = \mathbf{f}_{m_X} \otimes_S \mathbf{e}_{m_X} \mid 1 \leq j \leq r, 1 \leq i \leq s\}.$$

$$F_{ijj'l} = \sum_{h=1}^{j'-l+1} \mathbf{f}_{(i,j,h)} \otimes_k \mathbf{e}_{(i,j',l+h-1)}, \quad \begin{cases} l = 1, \dots, j', & \text{if } j > j'; \\ l = 2, \dots, j', & \text{if } j = j'; \\ l = j, \dots, j', & \text{if } j < j'. \end{cases}$$

Define a path algebra R_{XY} , a minimal algebra S and a R_{XY} -module K over S :

$$\begin{aligned} R_{XY} : X \xrightarrow{a_1} Y, \quad S = \prod_{i=1}^3 S_{Z_i}, \quad S_{Z_i} = k1_{Z_i}, i = 1, 2, 3; \\ K_X = k1_{Z_2} \oplus k1_{Z_1}, \quad K_Y = k1_{Z_3} \oplus k1_{Z_2}, \quad K(a_1) = \begin{pmatrix} 0 & 1_{Z_2} \\ 0 & 0 \end{pmatrix} : K_X \rightarrow K_Y. \end{aligned} \quad (2.1-4)$$

Let $\text{End}_{R_{XY}}(K)$ be over Σ , the S -quasi basis and S - S -quasi-basis are given respectively by

$$\begin{aligned} F_{Z_1} &= f_{z_1(X,2)} \otimes_S e_{z_1(X,2)}, \quad F_{Z_3} = f_{z_3(Y,1)} \otimes_S e_{z_3(Y,1)}, \\ F_{Z_2} &= (f_{z_2(X,1)} \otimes_S e_{z_2(X,1)}, f_{z_2(Y,2)} \otimes_S e_{z_2(Y,2)}); \\ F_{Z_2 Z_1} &= f_{z_2(X,1)} \otimes_k e_{z_1(X,2)}, \quad F_{Z_3 Z_2} = f_{z_3(Y,1)} \otimes_k e_{z_2(Y,2)}. \end{aligned}$$

Define R_{XY}, S, K below, then $\text{End}_{R_{XY}}(K)$ over Σ has S -quasi basis $\{F_Z\}$ of index 0, and the part of index 1 in $\text{End}_{R_{XY}}(K)$ is 0. Let $R_X = k[x, \phi_X(x)^{-1}]$, $R_Y = k1_Y$ or $k[y, \phi_Y(y)]$:

$$\begin{aligned} R_{XY} : x \bigcirc X \xrightarrow{a_1} Y, \text{ or } x \bigcirc X \xrightarrow{a_1} Y \bigcirc y; \quad S = z \bigcirc X; \\ K_X = S, K_Y = S, K(a_1) = (1_Z), K(x) = (z), \text{ or add } K(y) = (z) \text{ if } Y \in \mathcal{T}_1; \\ F_Z = (f_{z_X} \otimes_S e_{z_X}, f_{z_Y} \otimes_S e_{z_Y}). \end{aligned} \quad (2.1-5)$$

Definition 2.1.1 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair, let R' be a minimal algebra with algebra $\Delta' = \sum_{p=1}^{\infty} R'^{\otimes p}$ in Formula (1.1-3), and $\underline{d} = (n_X \mid X \in \mathcal{T})$ a dimension vector over \mathcal{T} . An R' - \bar{R} -bi-module L (or an \bar{R} -module over R') is said to be admissible, if L satisfies (a1)-(13) below.

(a1) There are three cases:

① $d_X = 1$, or 0 for any $X \in \mathcal{T}$. $\bar{R} = R$ or $R[a_1]$ with $\delta(a_1) = 0$;

② $R_X = k[x, \phi(x)^{-1}]$, $\bar{R} = R$, and $R' = S \times \prod_{Y \neq X} R_Y$, $L = K \oplus (\oplus_{Y \in \mathcal{T}, Y \neq X} R_Y)$ with S, K defined in Formula (2.1-3);

③ $\bar{R} = R_{XY} \times \prod_{U \in \mathcal{T}, U \neq X, Y} R_U$ with $X, Y \in \mathcal{T}_0$, $\delta(a_1) = 0$, and $R' = S \times \prod_{U \neq X, Y} R_U$, $L = K \oplus (\oplus_{U \in \mathcal{T}, U \neq X, Y} R_U)$ with S, K defined in Formula (2.1-4).

(a2) Denote unified by $L = \oplus_{X \in \mathcal{T}} L_X$, $L_X = \oplus_{p=1}^{n_X} R'_{Z(X,p)}$, $X \in \mathcal{T}$. Let $e_{z(X,p)}$ be a $(1 \times n_X)$ -matrix row with the p -th entry $1_{Z(X,p)}$ and others zero. Let $L^* = \text{Hom}_{R'}(L, R')$ be an \bar{R} - R' -bi-module, $f_{z(X,p)} = e_{z(X,p)}^* = e_{z(X,p)}^T$ be an $(n_X \times 1)$ -matrix column. And $e(f) = ef$.

(a3) $\bar{E} = \text{End}_{\bar{R}}(L) \subseteq \prod_{X \in \mathcal{T}} \mathbb{T}_{n_X}(\Delta')$, where $\bar{E}_0 \simeq R'$; \bar{E}_1 is a quasi-free R' - R' -bi-module; and \bar{E} is finitely generated in index $(0, 1)$. Forgotten the \bar{R} - \bar{R} -structure on $L^* \otimes L$, we assume

$$\begin{aligned} \bar{E}_0 &\subseteq \prod_{X \in \mathcal{T}} \mathbb{D}_{n_X}(R') \subseteq \prod_{X \in \mathcal{T}} (L^* \otimes_{R'} L); \\ \bar{E}_1 &\subseteq \prod_{X \in \mathcal{T}} \mathbb{N}_{n_X}(R' \otimes_k R') \subseteq \prod_{X \in \mathcal{T}} (L^* \otimes_k L). \end{aligned}$$

Lemma 2.1.2 Let D be a commutative algebra, and Λ, Σ be commutative D -algebras. Let ${}_{\Lambda}\mathcal{G}$ and ${}_{\Sigma}\mathcal{S}$ be finitely generated projective left Λ -module and right Σ -module respectively, then there exists a $\Lambda \otimes_D \Sigma$ -module isomorphism

$$\text{Hom}_{\Lambda}(\mathcal{G}, \Lambda) \otimes_D \text{Hom}_{\Sigma}(\mathcal{S}, \Sigma) \cong \text{Hom}_{\Lambda \otimes_D \Sigma}(\mathcal{G} \otimes_D \mathcal{S}, \Lambda \otimes_D \Sigma).$$

Proof We first claim, that $\mathcal{G} \otimes_D \mathcal{S}$ is a projective $\Lambda \otimes_D \Sigma$ -module. In fact, suppose $\mathcal{G} \oplus \mathcal{G}' = \mathcal{F}_1$, and $\mathcal{S} \oplus \mathcal{S}' = \mathcal{F}_2$ are free modules over Λ, Σ respectively. Then $\mathcal{G} \otimes_D \mathcal{S}$ is a direct summand of the free $\Lambda \otimes_D \Sigma$ -module $\mathcal{F}_1 \otimes \mathcal{F}_2$. Consequently, $\text{Hom}_{\Lambda \otimes_D \Sigma}(\mathcal{G} \otimes_D \mathcal{S}, \Lambda \otimes_D \Sigma)$ is also a projective $\Lambda \otimes_D \Sigma$ -module. Consider the following commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\Lambda}(\mathcal{G}, \Lambda) \times \text{Hom}_{\Sigma}(\mathcal{S}, \Sigma) & \longrightarrow & \text{Hom}_{\Lambda}(\mathcal{G}, \Lambda) \otimes_D \text{Hom}_{\Sigma}(\mathcal{S}, \Sigma) \\ \psi \downarrow & \swarrow \tilde{\psi} & \\ \text{Hom}_{\Lambda \otimes_D \Sigma}(\mathcal{G} \otimes_D \mathcal{S}, \Lambda \otimes_D \Sigma) & & \end{array}$$

Let $f \in \text{Hom}_\Lambda(\mathcal{G}, \Lambda)$ and $g \in \text{Hom}_\Sigma(\mathcal{S}, \Sigma)$. Since f and g are D -linear, there exists a $\Lambda \otimes_D \Sigma$ -linear map $\psi : \mathcal{G} \otimes_D \mathcal{S} \rightarrow \Lambda \otimes_D \Sigma$, such that $(\psi(f, g))(x \otimes y) = f(x) \otimes g(y)$, for $(x, y) \in \mathcal{G} \times \mathcal{S}$. Now $\psi(fr, g) = \psi(f, rg)$ for $r \in D$. Thus there exists a unique $(\Lambda \otimes_D \Sigma)$ -linear map $\tilde{\psi}$ given by $f \otimes g \mapsto \psi(f, g)$, which is clearly natural in both \mathcal{G} and \mathcal{S} . $\tilde{\psi}$ is an isomorphism if ${}_\Lambda \mathcal{G}, \mathcal{S}_\Sigma$ are free, consequently so is for ${}_\Lambda \mathcal{G}, \mathcal{S}_\Sigma$ being projective by [J] p134, Proposition 3.4-3.5. This completes the proof.

Proposition 2.1.3 If \bar{R}, R' are viewed as categories A', B' respectively, and L as a functor θ' , then θ' is an admissible functor given by Defined 4.3 of [CB1].

(We stress in particular, that the opposite construction is usually impossible. Throughout the paper, we use the right module structure and upper triangular matrix, which is opposite to the left module and lower triangular matrix used in [CB1].)

Proof (i) Let $\theta'(X) = \bigoplus_{p=1}^{n_X} Z_{(X,p)}$, (a1) implies (A1)-(A2) of Definition 4.3 in [CB1].

(ii) Let $\bar{E}_0^* = \text{Hom}_{R'}(\bar{E}_0, R')$, then $\bar{E}_0^* \simeq \text{Hom}_{R'}(R', R') \simeq R'$. And by Lemma 1.2.2,

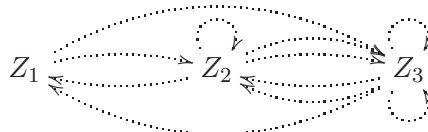
$$\begin{aligned} \text{Hom}_{R'}(L^* \otimes_{R'} L, R') &\simeq \text{Hom}_{R' \otimes_{R'} R'}(L^* \otimes_{R'} L, R' \otimes_{R'} R') \\ &\simeq \text{Hom}_{R'}(L^*, R') \otimes_{R'} \text{Hom}_{R'}(L, R') \simeq L \otimes_{R'} L^*. \end{aligned}$$

We may establish an equivalent relation \sim on the elements of $L \otimes_{R'} L^*$: two elements are equivalent, if and only if both of them acting on every R' -base matrix of \bar{E}_0 have the same value in R' . In the case of Definition 2.1.1 (a1) ①, if $\bar{R} = R[a_1]$ with R_{XY}, S, K given in Formula (2.1-5), $e_{Z_X} \otimes_{R'} f_{Z_X}$ and $e_{Z_Y} \otimes_{R'} f_{Z_Y}$ acting on F_Z equal 1_Z , and 0 on others; while $e_{Z_X} \otimes_{\bar{R}} f_{Z_X} = e_{Z_Y} \otimes_{\bar{R}} f_{Z_Y}$ by carrying a_1 across the tensor product. In (a1) ②, $e_{(ijl)} \otimes_{R'} f_{(ijl)}$ acting on \bar{F}_{ij} equal $1_{Z_{ij}}$, and 0 on others; while $e_{(ij1)} \otimes_{\bar{R}} f_{(ij1)} = \dots = e_{(ijj)} \otimes_{\bar{R}} f_{(ijj)}$. In (a1) ③, $e_{Z_2(X,1)} \otimes_{R'} f_{Z_2(X,1)}$ and $e_{Z_2(Y,2)} \otimes_{R'} f_{Z_2(Y,2)}$ acting on (F_{X,Z_2}, F_{Y,Z_2}) equal 1_{Z_2} , and 0 on others; while $e_{Z_2(X,1)} \otimes_{\bar{R}} f_{Z_2(X,1)} = e_{Z_2(Y,2)} \otimes_{\bar{R}} f_{Z_2(Y,2)}$. And in all the cases, we have a unique $e_{Z_Z} \otimes_{R'} f_{Z_Z}$ acting on $e_{Z_Z} \otimes_{R'} f_{Z_Z}$ equals 1_Z , and 0 on others. Moreover any element besides action on \bar{E}_0 is 0. Therefore $(L \otimes_{R'} L^* / \sim) \simeq L \otimes_{\bar{R}} L^*$, and we obtain the R' -quasi-basis of \bar{E}_0^* formed as $\{e_{Z_{X,p}} \otimes_{\bar{R}} f_{Z_{X,p}}\}$ dual to that of \bar{E}_0 .

(iii) Let $\bar{E}_1^* = \text{Hom}_{R'^{\otimes 2}}(\bar{E}_1, R'^{\otimes 2})$, by Lemma 2.1.2:

$$\text{Hom}_{R' \otimes_k R'}(L^* \otimes_k L, R' \otimes_k R') \simeq \text{Hom}_{R'}(L^*, R') \otimes_k \text{Hom}_{R'}(L, R') \simeq L \otimes_k L^*. \quad (2.1-6)$$

We establish an equivalent relation \sim on the elements of $L \otimes_k L^*$: two elements are equivalent, if and only if both of them acting on every base matrix of \bar{E}_1 have the same value in $R' \otimes_k R'$. In the case of Definition 2.1.1 (a1) ①, $\bar{E}_1 = 0$, so that $\bar{E}_1^* = 0$. In (a1) ②, $e_{i,j,h} \otimes_k f_{i,j',l+h-1}$ acting on $F_{ijj'l}$ equal $1_{Z_{ij}} \otimes_k 1_{Z_{ij'}}$, and 0 on others; while $e_{(ij1)} \otimes_{\bar{R}} f_{(ij'l)} = \dots = e_{(ij,j'-l+1)} \otimes_{\bar{R}} f_{(ij'j')}$, $\forall j \geq j'$ and $e_{(ij1)} \otimes_{\bar{R}} f_{(ij'l)} = \dots = e_{(ij,j-l+1)} \otimes_{\bar{R}} f_{(ij'j')} \forall j < j'$ by carrying x across the tensor product. In (a1) ③, we have the unique element $e_{Z_2(X,1)} \otimes_{R'} f_{Z_1(X,2)}$ acting on $F_{Z_2 Z_1}$ equals $1_{Z_2} \otimes_k 1_{Z_1}$, and 0 on $F_{Z_3 Z_2}$; similar for $e_{Z_3(Y,1)} \otimes_{R'} f_{Z_3(Y,2)}$. Moreover any element besides action on \bar{E} is 0. Therefore $(L \otimes_k L^* / \sim) \simeq L \otimes_{\bar{R}} L^*$, and we obtain the $R'-R'$ -quasi-basis of \bar{E}_1^* formed as $\{e_{Z_{X,p}} \otimes_{\bar{R}} f_{Z_{X,q}}\}$ dual to that of \bar{E}_1 . The picture below shows $e_{(ij1)} \otimes_{\bar{R}} f_{(ij'l)}$ in \bar{E}_1^* of (a2) as dotted arrows in the case of $s = 1, r = 3$:



Summary up (ii)-(iii), $B' \otimes_{A'} B'$ is determined by $\bar{E}_0^* \oplus \bar{E}_1^*$, J' by \bar{E}_1^* is projective. There is a natural map $(\bar{E}_0^* \oplus \bar{E}_1^*) \rightarrow R', e_{Z_{(X,p)}} \otimes_{\bar{R}} f_{Z_{Y,q}} \mapsto e_{Z_{(X,p)}} f_{Z_{(Y,q)}}$, which equals $1_{Z_{(X,p)}}$ for $Z_{(X,p)} = Z_{(Y,q)}$, or 0 otherwise. Thus J' is the kernel of the map $B' \otimes_{A'} B' \rightarrow B'$. (A3) follows.

(iv) (A4)-(A6) are easy. There is a natural ordering on the basis of \bar{E}_1^* transferred from that of \bar{E}_1 , (A7) follows. The lemma is proved.

More generally, \bar{E}_0 has a R' -quasi-basis $\{F_Z = (F_{Z,X} \mid X \in \mathcal{T}) \mid Z \in \mathcal{T}'\}$, such that

$$F_{Z,X} = \text{diag}(s_{Z_{(X,1)}}, \dots, s_{Z_{(X,n_X)}}), \quad \begin{cases} s_{Z_{(X,p)}} = 1_Z, & \text{for } Z_{(X,p)} = Z; \\ s_{Z_{(X,p)}} = 0, & \text{for } Z_{(X,p)} \neq Z. \end{cases}$$

And \bar{E}_1 has a R' - R' -quasi basis F_1, \dots, F_l , where for $i = 1, \dots, l$:

$$F_i = (F_{i,X} \mid X \in \mathcal{T}), \quad F_{i,X} \in \mathbb{N}_{n_X}(R' \otimes_k R').$$

The multiplication of two base matrices is given by usual Δ' -matrix product component wise.

$\text{IM}_{n_X \times n_Y}(R' \otimes_k R'), \forall X, Y \in \mathcal{T}$, possesses an \bar{R} - \bar{R} -bi-module structure as follows: taken any $f_{Z_{(X,p)}} \otimes_k e_{Z_{(Y,q)}}$, $b, c \in \{x \mid X \in \mathcal{T}\} \cup \{a_1\}$ with $e(b) = X, s(c) = Y$,

$$b \otimes_{\bar{R}} (f_{Z_{(X,p)}} \otimes_k e_{Z_{(Y,q)}}) \otimes_{\bar{R}} c = L(b)(f_{Z_{(X,p)}} \otimes_k e_{Z_{(Y,q)}})L(c).$$

Construction 2.1.4 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, d)$ be a matrix bi-module problem, and \mathfrak{B} the associated boc. Suppose $\bar{R}, R', L, \underline{d}$ are given in Definition 2.1.1. Then there is an induced matrix bi-module problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ in the following sense.

(i) The size vector of the matrices in $\mathcal{K}', \mathcal{M}'$ and H' over \mathcal{T} is \underline{n} determined by \underline{d} : $n_i = n_X$ for $i \in X$. Then $t' = \sum_{i \in T} n_i$, the set of integers $T' = \{1, \dots, t'\}$.

(ii) $\mathcal{K}'_0 \simeq R'$, an isomorphism $\bar{E}_0 \xrightarrow{\nu_0} \mathcal{K}'_0$ gives the R' -quasi-basis $\{E'_Z = \sum_{X \in \mathcal{T}} F_{Z,X} * E_X \in \mathbb{D}_{t'}(R') \mid Z' \in \mathcal{T}'\}$ of \mathcal{K}'_0 , where $*$ is given by Definition 1.3.2 (ii) for $S = R', p = 1$. $\mathcal{K}'_1 = \mathcal{K}'_{10} \oplus \mathcal{K}'_{11}$. An isomorphism $\bar{E} \xrightarrow{\nu_1} \mathcal{K}'_{10}$ gives the R' - R' -quasi basis $\mathcal{F}' = \{\sum_{X \in \mathcal{T}} F_{i,X} * E_X \mid i = 1, \dots, l\}$ of \mathcal{K}'_{10} given by 1.3.2 (ii) for $S = R', p = 2$; $\mathcal{K}'_{11} = (L^* \otimes_k L) \otimes_{R^{\otimes 2}} \mathcal{K}_1$ with a basis $\mathcal{U}' = \{(f_{Z_{(X'_j,p)}} \otimes_k e_{Z_{(Y'_j,q)}}) * V_j \mid 1_{X'_j} V_j 1_{Y'_j} = V_j, \forall p, q; j = 1, \dots, m\} \subseteq \mathbb{N}_{t'}(R' \otimes_k R')$ by 1.3.2

(iii) for $S = R', p = 2$. The basis of \mathcal{K}'_1 is $\mathcal{V}' = \mathcal{F}' \cup \mathcal{U}'$.

(iii) $\mathcal{M}'_1 \simeq (L^* \otimes_k L) \otimes_{R^{\otimes 2}} \mathcal{M}_1$, with the normalized R' - R' -quasi-basis $\mathcal{A}' = \{(f_{Z_{(X_i,p)}} \otimes_k e_{Z_{(Y_i,q)}}) * A_i \mid 1_{X_i} A_i 1_{Y_i} = A_i, \forall p, q\}$ given by Definition 1.3.2 (iii) for $S = R', p = 2$, where $1 \leq i \leq n$ if $\bar{R} = R$, and $1 < i \leq n$ if $\bar{R} = R[a_1]$.

(iv) $H' = \sum_{X \in \mathcal{T}} H_X(L_X(x)) + L(a_1) * A_1$, where $L_X(x) = \bar{W}_X$, $H_X(\bar{W}_X)$ is defined in 1.3.2 (i); and $*$ is given by 1.3.2 (iii) for $S = R', p = 1$.

The multiplication is given by usual Δ' -matrix product according to Lemma 1.3.3, as an example, we calculate $r'_{11} : \mathcal{M}'_1 \otimes \mathcal{K}'_1 \rightarrow \mathcal{K}'_2$ by substituting \bar{W}_X for x :

$$\begin{aligned} ((f_{X_l,p} \otimes_k e_{Z_{Y_l,q}}) * A_l)(F_{h,Y_l} * E_{Y_l}) &= ((f_{Z_{X_l,p}} \otimes_k e_{Z_{Y_l,q'}}) F_{h,Y_l}) * (A_l E_{Y_l}); \\ ((f_{Z_{X_i,p_1}} \otimes_k e_{Z_{Y_i,q_1}}) * A_i)((f_{Z_{X'_j,p_2}} \otimes_k e_{Z_{Y'_j,q_2}}) * V_j) \\ &= \sum_l ((f_{Z_{X_i,p_1}} \otimes_k e_{Z_{Y_i,q_1}})(f_{Z_{X'_j,p_2}} \otimes_k e_{Z_{Y'_j,q_2}}) \otimes_{R^{\otimes 3}} \sigma_{ijl}) * A_l, \end{aligned}$$

by 1.3.3 (ii) and (iii) for $p = 2 = q$. m'_{11}, l'_{11}, d'_1 are similar. The proof is finished.

Proposition 2.1.5 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair, and \mathfrak{A}' be given by Construction 2.1.4. Then the associated boc \mathfrak{B}' of \mathfrak{A}' is the induced boc of \mathfrak{B} given by Proposition 4.5 in [CB1]. And $R(\mathfrak{A}') \simeq R(\mathfrak{B}')$.

Proof Denote by $\mathcal{C}' = (R', \mathcal{C}', \mathcal{N}', \partial')$ the associated bi-co-module problem of \mathfrak{A}' .

(i) $\mathcal{C}'_0 = \text{Hom}_{R'}(\mathcal{K}'_0, R')$. The isomorphism $\bar{E}_0^* = \text{Hom}_{R'}(\bar{E}_0, R') \xrightarrow{\nu_0^*} \text{Hom}_{R'}(\mathcal{K}'_0, R') = \mathcal{C}'_0$ gives the R' -quasi-basis $\{e'_Z \mid Z \in \mathcal{T}'\}$ of \mathcal{C}'_0 , the image of the basis of \bar{E}_0^* given in the proof (ii) of Proposition 2.1.3 under ν_0^* , which is R' -dual to that of \mathcal{K}'_0 .

(ii) $\mathcal{C}'_1 = \text{Hom}_{R'^{\otimes 2}}(\mathcal{K}', R'^{\otimes 2}) \simeq \mathcal{C}'_{10} \oplus \mathcal{C}'_{11}$. $\bar{E}_1^* = \text{Hom}_{R^{\otimes 2}}(\bar{E}_1, R^{\otimes 2}) \xrightarrow{\nu_1^*} \text{Hom}_{R'^{\otimes 2}}(\mathcal{K}'_{10}, R'^{\otimes 2}) = \mathcal{C}'_{10}$ is an isomorphism, the R' - R' -quasi-basis of \mathcal{C}'_{10} is the image of that in \bar{E}_1^* given by the proof (iii) of 2.1.3 under ν_1^* . According to Lemma 2.1.2 and Formula (2.1-6):

$$\begin{aligned} \mathcal{C}'_{11} &= \text{Hom}_{R'^{\otimes 2}}(\mathcal{K}'_{11}, R'^{\otimes 2}) = \text{Hom}_{R'^{\otimes 2}}((L^* \otimes_k L) \otimes_{R^{\otimes 2}} \mathcal{K}_1, R'^{\otimes 2}) \\ &\simeq \text{Hom}_{R'^{\otimes 2} \otimes_{R^{\otimes 2}} R^{\otimes 2}}((L^* \otimes_k L) \otimes_{R^{\otimes 2}} \mathcal{K}_1, R'^{\otimes 2} \otimes_{R^{\otimes 2}} R^{\otimes 2}) \\ &\simeq \text{Hom}_{R'^{\otimes 2}}(L^* \otimes_k L, R'^{\otimes 2}) \otimes_{R^{\otimes 2}} (\mathcal{K}_1, R^{\otimes 2}) \simeq (L \otimes_k L^*) \otimes_{R^{\otimes 2}} \mathcal{C}_1. \end{aligned}$$

Write $\mathcal{U}'^* = \{(e_{Z_{(X'_j, p)}} \otimes_k f_{Z_{(Y'_j, q)}}) \otimes_{R^{\otimes 2}} v_j \mid \forall p, q; 1 \leq j \leq m\}$, which is R' - R' -dual to \mathcal{U}' given in Construction 2.1.4 (ii). The R' - R' -quasi basis \mathcal{V}'^* of \mathcal{C}'_1 is given respectively by: $\mathcal{V}'^* = \mathcal{U}'^*$ in the case of Definition 2.1.1 (a1) ①; $\mathcal{V}'^* = \mathcal{U}'^* \cup \{e_{(ij1)} \otimes_{\bar{R}} f_{(ij'l)} \mid i, j, l\}$ in (a1) ②; $\mathcal{V}'^* = \mathcal{U}'^* \cup \{e_{Z_{1(X, 2)}} \otimes_{\bar{R}} f_{Z_{1(X, 2)}}, e_{Z_{3(Y, 1)}} \otimes_{\bar{R}} f_{Z_{3(Y, 1)}}\}$ in (a1) ③, which is dual to the basis \mathcal{V}' of \mathcal{K}'_1 .

(iii) $\mathcal{N}'_1 = \text{Hom}_{R'^{\otimes 2}}(\mathcal{M}'_1, R'^{\otimes 2}) \simeq (L \otimes_k L^*) \otimes_{R^{\otimes 2}} \mathcal{M}_1$, with R' - R' -quasi basis $\mathcal{A}'^* = \{(e_{Z_{(X_i, p)}} \otimes_k f_{Z_{(Y_i, q)}}) \otimes_{R^{\otimes 2}} a_i \mid \forall p, q\}$ dual to \mathcal{A}' , where $1 \leq i \leq n$ for $\bar{R} = R$, and $1 < i \leq n$ for $\bar{R} = R[a_1]$.

(iv) The co-multiplication μ' , the left, (resp. right) co-module action ι' , (resp. τ'), and the co-derivation ∂ are dual to m', l', r', d' respectively. For example, $\tau'_{11} : \mathcal{N}'_2 \rightarrow \mathcal{N}'_1 \otimes_{R'} \mathcal{C}'_1$,

$$\begin{aligned} \tau'_{11}((e_{Z_{X_l, p_1}} \otimes_k f_{Z_{Y_l, q_2}}) \otimes_{R^{\otimes 2}} a_l) &= \sum_{q > q_2} ((e_{Z_{X_l, p_1}} \otimes_R a_l \otimes_R f_{Z_{Y_l, q}}) \otimes_{R'} (e_{Z_{Y_l, q}} \otimes_{\bar{R}} f_{Z_{Y_l, q_2}})) \\ &+ \sum_{i, j} (e_{Z_{X_l, p_1}} \otimes_R a_i \otimes_R (\sum_{X=c(a_i)=s(v_j); p, q} f_{Z_{X, p}} \otimes_R e_{Z_{X, q}}) \otimes_R v_j \otimes_R f_{Z_{Y_l, q_2}}) \otimes_{R^{\otimes 3}} \sigma_{ijl}. \end{aligned}$$

Finally, we obtain the boc \mathfrak{B}' of \mathfrak{C}' , with a layered L' and the differential δ' given in [CB1] 4.5, the proof is finished.

2.2 Reductions for matrix bi-module problems

The present subsection is devoted to introducing seven reductions of matrix bi-module problems based on Construction 2.1.4, where the last two do not occur in the previous papers on bocses. And finally we give a regularization as the eighth reduction.

Proposition 2.2.1 (Localization) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with $R_X = k[x, \phi(x)^{-1}]$, and $R'_X = k[x, \phi(x)^{-1}c(x)^{-1}]$ a finitely generated localization of R_X . Define $\bar{R} = R$, the minimal algebra $R' = R'_X \times \prod_{Y \in \mathcal{T} \setminus \{X\}} R_Y$, $L = R'$. Then L is an admissible R' - \bar{R} -bi-module.

(i) There exists an induced matrix bi-module problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ of \mathfrak{A} and a fully faithful functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced boc \mathfrak{B}' of \mathfrak{B} given by localization [CB1, 4.8] is the associated boc of \mathfrak{A}' .

Proposition 2.2.2 (Loop mutation) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair, $X \in \mathcal{T}_0, a_1 : X \mapsto X, \delta(a_1) = 0$. Define $\bar{R} = R[a_1]$, a minimal algebra $R' = R'_X \times \prod_{Y \in \mathcal{T} \setminus \{X\}} R_Y$, with $R'_X = k[x]$, and $L = R'$. Then L is an admissible R' - \bar{R} -bi-module.

(i) There exists an induced matrix bi-module problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', d')$ of \mathfrak{A} , and an equivalent functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced boc \mathfrak{B}' of \mathfrak{B} given by $\theta' : A' \rightarrow B'$, with $\theta'(Y) = Y, \forall Y \in \mathcal{T}, \theta'(a_1) = x$, is the associated boc of \mathfrak{A}' by Proposition 2.1.3.

Proposition 2.2.3 (Deletion) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair, $\mathcal{T}' \subset \mathcal{T}$. Define $\bar{R} = R, R' = \prod_{X \in \mathcal{T}'} R_X$, and $L = R'$. Then L is an admissible R' - \bar{R} -bi-module.

(i) There exists an induced matrix bi-module problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ of \mathfrak{A} , and a fully faithful functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced boc \mathfrak{B}' obtained by deletion of $\mathcal{T} \setminus \mathcal{T}'$ given by [CB1, 4.6] is the associated boc of \mathfrak{A}' .

Proposition 2.2.4 (Unraveling) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with $R_X = k[x, \phi(x)^{-1}]$. Define $\bar{R} = R$, $R' = S \times \prod_{Z \in \mathcal{T} \setminus \{X\}} R_Z$, $L = K \oplus (\oplus_{Z \in \mathcal{T} \setminus \{X\}} R_Z)$ with S and K given by Formula (2.1-3). Then L is an admissible R' - \bar{R} -bi-module according to Definition 2.1.1 (a1) ②.

(i) There exists an induced matrix bi-module problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ and a fully faithful functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced boc \mathfrak{B}' given by unraveling in [CB1, 4.7] is the associated boc of \mathfrak{A}' .

Proposition 2.2.5 (Edge reduction) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair, $X, Y \in \mathcal{T}_0$, $a_1 : X \mapsto Y$, $\delta(a_1) = 0$. Define $\bar{R} = R[a_1]$, $R' = S \times \prod_{Z \in \mathcal{T} \setminus \{X, Y\}} R_Z$, $L = K \oplus (\oplus_{Z \in \mathcal{T} \setminus \{X, Y\}} R_Z)$ with S and K defined in Formula (2.1-4). Then L is an admissible R' - \bar{R} -bi-module by Definition 2.1.1 (a1) ③.

(i) There exists an induced matrix bi-module problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$, and an equivalence $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced boc \mathfrak{B}' given by edge reduction in [CB1, 4.9] is the associated boc of \mathfrak{A}' .

Proposition 2.2.6 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair, $X, Y \in \mathcal{T}_0$, $a_1 : X \mapsto Y$, $\delta(a_1) = 0$. Set $\bar{R} = R[a_1]$, $R' = R$, $L = K \oplus (\oplus_{U \in \mathcal{T} \setminus \{X, Y\}} R_U)$ with $K : R_X \xrightarrow{(0)} R_Y$. Then L is an admissible R' - \bar{R} -bi-module.

(i) There is an induced problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', d')$, with $\mathcal{K}' = \mathcal{K}$; $\mathcal{M}' = \mathcal{M}^{(1)}$; $H' = H$. And an induced fully faithful functors $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$. The subcategory of $R(\mathfrak{A})$ consisting of representations P with $P(a_1) = 0$ is equivalent to $R(\mathfrak{A}')$.

(ii) The induced boc \mathfrak{B}' given by the admissible functor $\theta' : A' \rightarrow B'$ with $\theta'(U) = U, \forall U \in \mathcal{T}$ and $\theta'(a_1) = 0$ is the associated boc of \mathfrak{A}' .

Proposition 2.2.7 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair, $X, Y \in \mathcal{T}_0$, $a_1 : X \mapsto Y$, $\delta(a_1) = 0$. Set $\bar{R} = R[a_1]$, $R' = S \times \prod_{U \in \mathcal{T} \setminus \{X, Y\}} R_U$, $L = K \oplus (\oplus_{U \in \mathcal{T} \setminus \{X, Y\}} R_U)$ with S and K defined in Formula (2.1-5). Then L is an admissible R' - \bar{R} -bi-module.

(i) There is an induced local problem \mathfrak{A}' , and an induced fully faithful functors $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$. The subcategory of $R(\mathfrak{A})$ consisting of P of size vector \underline{m} with $m_X = m_Y$, and $\text{rank}(P(a_1)) = m_X$ for $Y \in \mathcal{T}_0$, or $P(a_1)^{-1}P(x)P(a_1) = P(y)$ for $Y \in \mathcal{T}_1$, is equivalent to $R(\mathfrak{A}')$.

(ii) The induced boc \mathfrak{B}' given by the admissible functor $\theta' : A' \rightarrow B'$ with $\theta'(X) = Z = \theta'(Y)$; $\theta'(x) = z, \theta'(a_1) = (1)$, or in addition $\theta'(y) = z$ if $Y \in \mathcal{T}_1$, is the associated boc of \mathfrak{A}' .

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, d)$ be a matrix bi-module problem, $\mathfrak{C} = (R, \mathcal{C}, \mathcal{N}, \partial)$ be the associated bi-co-module problem, and \mathfrak{B} the boc of \mathfrak{C} . Then

$$\begin{aligned} d(V_1) &= A_1 + \sum_{l>1} \zeta_{1l} A_l, \quad d(V_j) \in \mathcal{M}_1^{(1)}, j \geq 2 \text{ in } \mathfrak{A} \\ &\iff \partial(a_1) = v_1 \text{ in } \mathfrak{C} \iff \delta(a_1) = v_1 \text{ in } \mathfrak{B}. \end{aligned} \quad (2.2-1)$$

In fact, since $\partial(a_1) = \sum_{j=1}^m \zeta_{j1} v_j$, we have $\partial(a_1) = v_1$, if and only if $\zeta_{11} = 1_{s(a_1)} \otimes_k 1_{t(a_1)}$, and $\zeta_{j1} = 0$ for all $j \geq 2$, if and only if $d(V_1) = A_1 + \sum_{l>1} \zeta_{1l} A_l$ and $d(V_j) \in \mathcal{M}^{(2)}$ for all $j \geq 2$, since $d(V_j) = \zeta_{j1} A_1 + \sum_{i>1} \zeta_{ji} A_i$. Finally, since $\iota_1(a_1) = 0, \tau_1(a_1) = 0$ by triangularity of \mathfrak{C} , $\delta(a_1) = v_1$ in \mathfrak{B} , if and only if $\partial(a_1) = v_1$ in \mathfrak{C} by Theorem 1.4.2.

Remark Let $\mathfrak{A}, \mathfrak{C}$ be given above, with $\partial(a_1) = v_1$, then

(i) $\mathcal{K}^{(1)} = \mathcal{K}_0 \oplus (\oplus_{j=2}^m \bar{\Delta} \otimes_{R^{\otimes 2}} V_j)$ is a sub-algebra of \mathcal{K} , and $\mathcal{M}^{(1)} = \oplus_{i=2}^n \bar{\Delta} \otimes_{R^{\otimes 2}} a_i$ is a $\mathcal{K}^{(1)}$ - $\mathcal{K}^{(1)}$ -sub-bi-module;

(ii) $\mathcal{C}^{(1)} = \bar{\Delta} \otimes_{R^{\otimes 2}} v_1$ is a co-ideal of \mathcal{C} , $\mathcal{C}^{[1]} = \mathcal{C}/\mathcal{C}^{(1)}$ is a quotient co-algebra, and $\mathcal{N}^{(1)} = \bar{\Delta} \otimes_{R^{\otimes 2}} a_1$ is a \mathcal{C} - \mathcal{C} -sub-bi-co-module, thus $\mathcal{N}^{[1]} = \mathcal{N}/\mathcal{N}^{(1)}$ is a $\mathcal{C}^{[1]}$ - $\mathcal{C}^{[1]}$ -quotient bi-co-module.

Proof (i) For any $V_i, V_j \in \mathcal{K}_1$,

$$d(V_i V_j) = d(\sum_{l=1}^m \gamma_{ijl} \otimes_{R^{\otimes 2}} V_l) = \sum_{l=1}^m \gamma_{ijl} \otimes_{R^{\otimes 2}} d(V_l) = \sum_{l,p} (\gamma_{ijl} \otimes_{R^{\otimes 2}} \zeta_{lp}) \otimes_{R^{\otimes 2}} A_p.$$

By triangularity (2.1-1), $d(V_i V_j) = d(V_i) V_j + V_i d(V_j) \in \mathcal{M}^{(1)}$. Consequently, the coefficient of A_1 in the formula above $\sum_l (\gamma_{ijl} \otimes_{R^{\otimes 2}} \zeta_{l1}) = 0$, where $\zeta_{11} = 1, \zeta_{l1} = 0$ for $l > 1$ by the hypothesis,

so that $\gamma_{ij1} = 0$ for all $1 \leq i, j \leq m$. Therefore $V_i V_j = \sum_{l>1} \gamma_{ijl} \otimes_{R^{\otimes 2}} V_l \in \mathcal{K}^{(1)}$ and hence $\mathcal{K}^{(1)}$ is a subalgebra of \mathcal{K} . Finally, $\mathcal{M}^{(1)}$ is a $\mathcal{K}^{(1)}$ - $\mathcal{K}^{(1)}$ -bi-module still by triangularity.

(ii) Since $\mu(v_1) = \mu(\partial(a_1)) = (\partial \otimes \mathbb{1})(\iota(a_1)) + (\mathbb{1} \otimes \partial)(\tau(a_1)) \stackrel{(2.1-2)}{=} (\partial \otimes \mathbb{1})(e_{s(a_1)} \otimes_R a_1) + (\mathbb{1} \otimes \partial)(a_1 \otimes_R e_{t(a_1)}) = 0$, $\mathcal{C}^{(1)}$ is a co-ideal of \mathcal{C} , which finishes the proof.

Proposition 2.2.8 (Regularization) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with $\delta(a_1) = v_1$.

(i) There is an induced matrix bi-module problem $\mathfrak{A}' = (R, \mathcal{K}^{(1)}, \mathcal{M}^{(1)}, H)$ of \mathfrak{A} , and an equivalent functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced boc \mathfrak{B}' given by regularization [CB1, 4.2] is the associated boc of \mathfrak{A}' .

Proof (i) \mathfrak{A}' is a matrix bi-module problem by Formula (2.2-1) and Remark (i) above. Note that $R' = R, \mathcal{T}' = \mathcal{T}$, for any $P \in R(\mathfrak{A})$ of size vector \underline{m} , let $f = \sum_{X \in \mathcal{T}} I_{m_X} * E_X + P(a_1) * V_1$, then $P' = f^{-1} P f \in R(\mathfrak{A}')$. Therefore, ϑ is an equivalent functor.

(ii) $\mathfrak{C}' = (R, \mathcal{C}^{[1]}, \mathcal{M}^{[1]}, \bar{\partial})$ with $\bar{\partial}$ induced from ∂ is the associated bi-co-module problem of \mathfrak{A}' by Remark (ii) above. Thus the associated boc \mathfrak{B}' is given by regularization from \mathfrak{B} .

Let $(\mathfrak{A}, \mathfrak{B})$ be a pair, with a layer $L = (R; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$ in \mathfrak{B} . Suppose $a_1 : X \mapsto Y$, $\delta(a_1) = \sum_{j=1}^m f_j(x, y) v_j \neq 0$. In order to obtain $\delta(a_1) = h(x, y) v'_1$, we make the following base change:

$$(v'_1, \dots, v'_m) = (v_1, \dots, v_m) F(x, y) \quad (2.2-2)$$

with $F(x, y) \in \text{IM}(R \otimes_k R)$ invertible. When $X \in \mathcal{T}_0$ or $Y \in \mathcal{T}_0$, R is preserved; but when $X, Y \in \mathcal{T}_1$, some localization $R'_X = R_X[c(x)^{-1}]$ (resp. $R'_Y = R_Y[c(y)^{-1}]$) is needed, see [CB1, §5]. Consequently, we have a base change of \mathcal{K}_1 dually given by

$$(V'_1, \dots, V'_m) = (V_1, \dots, V_m) F(x, y)^{-T}. \quad (2.2-3)$$

Finally we mention a simple fact according to all the reductions defined above. Suppose we start from a matrix bi-module problem $\mathfrak{A}^0 = (R^0, \mathcal{K}^0, \mathcal{M}^0, H = 0)$ with \mathcal{T}^0 trivial, if there is a sequence of reductions $\mathfrak{A}^0, \mathfrak{A}^1, \dots, \mathfrak{A}^r$ with $\mathfrak{A}^r = (R^r, \mathcal{K}^r, \mathcal{M}^r, H^r)$, and $X \in \mathcal{T}_1^r$, $H_X^r = (h_{ij}(x))$ of size t^r , then $h_{ij}(x) = a_{ij} + b_{ij}x \in k[x]$ is of degree 1.

2.3 Canonical Forms

We will give a canonical form (cf.[S]) for each representation of a matrix bi-module problem, and a sequence of reductions in the subsection.

Convention 2.3.1 Suppose \mathfrak{A} is a matrix bi-module problem, \mathfrak{A}' an induced problem and $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$ an induced functor. Let \underline{m}' be a size vector over \mathcal{T}' of \mathfrak{A}' , define a size vector $\underline{m} = (m_1, m_2, \dots, m_t)$ over \mathcal{T} of \mathfrak{A} based on \underline{m}' :

- (i) for regularization, loop mutation, localization, and Proposition 2.2.6, set $\underline{m} = \underline{m}'$;
- (ii) for deletion, set $m_i = m'_i$ if $i \in X, X \in \mathcal{T}'$, and 0 if $i \in X, X \in \mathcal{T} \setminus \mathcal{T}'$;
- (iii) for edge reduction, set $m_i = m'_i$ if $i \in Z, Z \neq X, Y$, $m_i = m'_{Z_1} + m'_{Z_2}$ if $i \in X$, and $m_i = m'_{Z_2} + m'_{Z_3}$ if $i \in Y$; for proposition 2.2.7, set $m_X = m'_Z = m_Y$;
- (iv) for unraveling, set $m_i = m'_i$ if $i \notin X$, and $m_i = \sum_{i=1}^s \sum_{j=1}^r i m'_{Z_{ij}} + m'_{Z_0}$ if $i \in X$.

Then \underline{m} is said to be the *size vector determined by \underline{m}'* , and is denoted by $\vartheta(\underline{m}')$.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bi-module problem with trivial \mathcal{T} , and \underline{m} be a size vector. For the sake of simplicity, write

$$H_{\underline{m}}(k) = \sum_{X \in \mathcal{T}} H_X(I_{m_X}), \quad H(k) = \sum_{X \in \mathcal{T}} H_X(1). \quad (*)$$

Note that if a size vector \underline{m} over \mathcal{T} is not sincere, let $\mathcal{T}' = \{i \mid m_i \neq 0\}$, set the induced bi-module problem \mathfrak{A}' given by a deletion of $\mathcal{T} \setminus \mathcal{T}'$, then the size vector $\underline{m}' = (m_i \mid m_i \neq 0)$ is sincere over \mathcal{T}' .

Lemma 2.3.2 (cf.[S]) Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bi-module problem with \mathcal{T} being trivial. Let P be a representation of \mathfrak{A} with a sincere size vector \underline{m} , by Definition 1.3.4,

$$P = H_{\underline{m}}(k) + \sum_{i=1}^n P(a_i) * A_i.$$

Then there exists a matrix bi-module problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ induced by one of the following three compositions:

- (i) Regularization;
- (ii) Edge reduction: edge reduction + deletion;
- (iii) Loop reduction: loop mutation + unraveling + deletion,

with a fully faithful functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$, such that there is a representation $P' \in R(\mathfrak{A}')$ having a sincere size vector \underline{m}' over \mathcal{T}' with $P \simeq \vartheta(P')$ and $\vartheta(\underline{m}') = \underline{m}$, where

$$P' = H_{\underline{m}'}(k) + B * A_1 + \sum_{i=1}^{n'} P(a'_i) * A'_i$$

as a k -matrix, B is given by one of Formulae (2.3-1)-(2.3-3) below.

Proof Let $a_1 : X \mapsto Y$ in the associated boc \mathfrak{B} of \mathfrak{A} .

- (i) If $\delta(a_1) = v_1$, denote by \emptyset a distinguished zero matrix-block. Set

$$B = \emptyset_{m_X \times m_Y}; \quad G = \emptyset_{1 \times 1}. \quad (2.3-1)$$

We proceed with a regularization for \mathfrak{A} , obtain an induced problem \mathfrak{A}' , and an equivalence $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$. Then $P \simeq \vartheta(P')$, P' is given in the proof (i) of Proposition 2.2.8.

- (ii) If $\delta(a_1) = 0$ and $X \neq Y$, we proceed with an edge reduction for \mathfrak{A} and obtain an induced problem \mathfrak{A}_1 . Let the invertible matrices $f_X \in \mathbb{M}_{m_X}(k)$, $f_Y \in \mathbb{M}_{m_Y}(k)$, such that

$$\begin{aligned} B &= f_X^{-1} P(a_1) f_Y = \begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix}_{m_X \times m_Y} \quad \text{with } r = \text{rank}(P(a_1)); \\ G &= (0), (1_{Z_2}), (0 \ 1_{Z_2}), \begin{pmatrix} 1_{Z_2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1_{Z_2} \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (2.3-2)$$

where the first four cases of G are obtained by a deletion for \mathfrak{A}_1 : ① $r = 0$, delete Z_2 ; now suppose $r > 0$, ② $m_X = r = m_Y$, delete Z_1 and Z_3 ; ③ $m_X = r$, $m_Y > r$, delete Z_1 ; ④ $m_X > r$, $m_Y = r$, delete Z_3 . We obtain an induced problem \mathfrak{A}' of \mathfrak{A}_1 , and a fully faithful functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$. Let $\underline{m}' = (m'_i)_{i \in \mathcal{T}'}$ be a size vector over \mathcal{T}' , with $m'_Z = m_Z$ for $Z \in \mathcal{T} \setminus \{X, Y\}$, $m'_{Z_1} = m_X - r$, $m'_{Z_2} = r$ and $m'_{Z_3} = m_Y - r$. Then there is some $P' = f^{-1} P f \in R(\mathfrak{A}')$ of size \underline{m}' with $P \simeq \vartheta(P')$ in $R(\mathfrak{A})$, where $f = f_X * E_X + f_Y * E_Y + \sum_{Z \in \mathcal{T} \setminus \{X, Y\}} I_{m_Z} * E_Z$.

- (iii) If $\delta(a_1) = 0$ and $X = Y$, suppose $P(a_1) \simeq J = \oplus_{i=1}^s (\oplus_j J_j(\lambda_i)^{e_{ij}})$, a Jordan form over k with the maximal size r of the Jordan blocks. We first proceed with a loop mutation $a_1 \mapsto (x)$, then with an unraveling for the polynomial $g(x) = (x - \lambda_1) \cdots (x - \lambda_s)$ and the positive integer r , thus obtain an induced problem \mathfrak{A}_1 of \mathfrak{A} . Let the invertible matrix $f_X \in \mathbb{M}_{m_X}(k)$, such that

$$B = f_X^{-1} P(a_1) f_X = W; \quad G = \bar{W}, \quad (2.3-3)$$

where W is a Weyr matrix, and $G = \bar{W}$ being a Weyr matrix similar to $\oplus_{e_{ij} \neq 0} J_j(\lambda_i) 1_{Z_{ij}}$ over S . Finally, delete a set of vertices $\{Z_0; Z_{ij} \mid e_{ij} = 0\}$ from \mathfrak{A}_1 . We obtain an induced problem \mathfrak{A}' of \mathfrak{A}_1 , and a fully faithful functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$. Let $\underline{m}' = (m'_i)_{i \in \mathcal{T}'}$ be a size vector over \mathcal{T}' with $m'_Z = m_Z$ for $Z \in \mathcal{T} \setminus \{X\}$, and $m'_{Z_{ij}} = e_{ij} \neq 0$; let $f = f_X * E_X + \sum_{Z \in \mathcal{T} \setminus \{X\}} I_{m_Z} * E_Z$, then $P' = f^{-1} P f \in R(\mathfrak{A}')$ with a sincere size vector \underline{m}' , such that $P \simeq \vartheta(P')$ in $R(\mathfrak{A})$.

In all the cases, $P' = H_{\underline{m}'}(k) + B * A_1 + \sum_{i=1}^{n'} P(a'_i) * A'_i$. The proof is completed.

Repeating the procedure of Lemma 2.3.2, we obtain the following theorem by induction.

Theorem 2.3.3 (cf.[S]) Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a matrix bi-module problem with \mathcal{T} trivial. Let \underline{m} be a sincere size vector over \mathcal{T} and $P \in R(\mathfrak{A})$ be a representation of size \underline{m} .

Then there exist a unique sequence of matrix bi-module problems, and a unique sequence of representations:

$$\begin{aligned} \mathfrak{A} &= \mathfrak{A}^0, \quad \mathfrak{A}^1, \quad \dots, \quad \mathfrak{A}^i, \quad \mathfrak{A}^{i+1}, \quad \dots, \quad \mathfrak{A}^r \\ G^0 &= 0, \quad G^1, \quad \dots, \quad G^i, \quad G^{i+1}, \quad \dots, \quad G^r \\ B^0 &= 0, \quad B^1, \quad \dots, \quad B^i, \quad B^{i+1}, \quad \dots, \quad B^r, \\ P^0 &= P, \quad P^1, \quad \dots, \quad P^i, \quad P^{i+1}, \quad \dots, \quad P^r \end{aligned}$$

where \mathfrak{A}^{i+1} is obtained from \mathfrak{A}^i according to one of the three reductions given in Lemma 2.3.2 for $i = 0, \dots, r-1$, such that

- (i) B^{i+1} and G^{i+1} is defined by one formula of (2.3-1)-(2.3-3).
- (ii) Let $\vartheta^{i,i+1} : R(\mathfrak{A}^{i+1}) \rightarrow R(\mathfrak{A}^i)$ be the induced functor, there is a sincere size vector \underline{m}^{i+1} over \mathcal{T}^{i+1} with $\vartheta^{i,i+1}(\underline{m}^{i+1}) = \underline{m}^i$ and some $P^{i+1} \in R(\mathfrak{A}^i)$ of size \underline{m}^{i+1} with $\vartheta^{i,i+1}(P^{i+1}) \simeq P^i$. Denote by A_1^j the first quasi-base matrix of \mathcal{M}_1^j , then $P^{i+1} = H_{\underline{m}^{i+1}}(k) + \sum_{j=1}^{n^{i+1}} M^{i+1}(a_j) * A_j^{i+1}$.
- (iii) Denote for $i < j$ the composition of induced functors $\vartheta^{ij} = \vartheta^{i,i+1} \dots \vartheta^{j-1,j} : R(\mathfrak{A}^j) \rightarrow R(\mathfrak{A}^i)$. Then $\vartheta^{0,i+1}(H_{\underline{m}^{i+1}}(k)) = \sum_{j=0}^i B^{j+1} * A_1^j \in R(\mathfrak{A})$.

With the notation of Theorem 2.3.3, if \mathfrak{A}^r is minimal:

$$\vartheta^{0r}(P^r) = \vartheta^{0r}(H_{\underline{m}^r}(k)) = \sum_{i=0}^{r-1} B^{i+1} * A_1^i \simeq P \in R(\mathfrak{A}). \quad (2.3-4)$$

The matrix $\vartheta^{0r}(P^r)$ is called the *canonical form* of P , and denoted by P^∞ .

In the second and the third sequence of Theorem 2.3.3, if G^{i+1} is obtained by an edge or loop reduction, then “1” appearing in B^{i+1} , which is not an eigenvalue in the case of B^{i+1} being a Weyr matrix, is called a *link* of P^∞ . And denote by $l(P^\infty)$ the number of links in P^∞ .

Corollary 2.3.4 [S, XZ] The canonical form of any representation P over a matrix bi-module problem $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ with R trivial is uniquely determined. Moreover,

- (i) for any $P, Q \in R(\mathfrak{A})$, $P \simeq Q$ if and only if P and Q have the same canonical form;
- (ii) P is indecomposable if and only if $l(P^\infty) = \dim(P) - 1$.

Corollary 2.3.5 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a matrix bi-module problem with \mathcal{T} trivial, let $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ be an induced problem obtained by a sequence of reductions, and $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$ be the induced functor. If \mathcal{T}' is trivial, then there is a unique reduction sequence $\mathfrak{A} = \mathfrak{A}^0, \mathfrak{A}^1, \dots, \mathfrak{A}^i, \mathfrak{A}^{i+1}, \dots, \mathfrak{A}^r = \mathfrak{A}'$ in the sense of Theorem 2.3.2 performed for $P = \vartheta(H'(k)) \in R(\mathfrak{A})$ according to Theorem 2.3.3.

Under the hypothesis of Corollary 2.3.5, let $\underline{m}^r = (1, \dots, 1)$ and $\underline{m}^i = \vartheta^{i,r}(\underline{m}^r)$, we give a R^r -structure on B^{i+1} in Theorem 2.3.3 and denoted by $G_r^{i+1} \in \text{IM}_{m_{s(a_1^i)}^i \times m_{t(a_1^i)}^i}(R^r)$: a non-zero element g_{pq} of $G_r^{i+1} * A_1^r$ belongs to $R_{X^r}^r$, whenever the entrance of the identity matrix $E^r \in \mathcal{K}^r$ at the p -th row, as well as the q -th column by the definition of H^r , is $1_{X^r} \in R_{X^r}^r$. Then

$$H^r = \sum_{i=0}^{r-1} G_r^{i+1} * A_1^i. \quad (2.3-5)$$

It is easy to see, that \mathfrak{A}^r is local if and only if $l(\vartheta^{0r}(H^r(k))) = \dim(\vartheta^{0r}(H^r(k))) - 1$. The non-eigenvalue 1_{X^r} appearing in G_r^i is called a link of H^r .

2.4 Defining systems

We introduce a concept of defining system in the subsection.

Let $B = (b_{ij})_{t \times t}$ and $C = (c_{ij})_{t \times t}$ be two $t \times t$ matrices over k . Given $1 \leq p, q \leq t$, the notation $B \equiv_{\prec(p,q)} C$ (resp. $B \equiv_{(p,q)} C$, $B \equiv_{\preceq(p,q)} C$) means that $b_{ij} = c_{ij}$ for any $(i, j) \prec (p, q)$ (resp. $(i, j) = (p, q)$, $(i, j) \preceq (p, q)$). One can define the similar notions for partitioned matrices.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a matrix bi-module problem with the associated boc \mathfrak{B} , where \mathcal{T} and $\mathcal{V} = \{V_1, \dots, V_m\}$ are both trivial. Suppose there exists a sequence with each reduction being in the sense of Lemma 2.3.2:

$$(\mathfrak{A}, \mathfrak{B}) = (\mathfrak{A}^0, \mathfrak{B}^0), (\mathfrak{A}^1, \mathfrak{B}^1) \dots, (\mathfrak{A}^i, \mathfrak{B}^i)(\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1}) \dots, (\mathfrak{A}^r, \mathfrak{B}^r), \dots, (\mathfrak{A}^s, \mathfrak{B}^s). \quad (2.4-1)$$

Sometimes, it is difficult to determine the dotted arrows in the induced boc after some reductions. Instead, we may consider a system of equations on dotted elements as variables (not dotted arrows), and give explicitly the linear relations on those elements (used in section 4.5).

Theorem 2.4.1 With the assumption above, for $i = 0, \dots, s$, there exists a matrix equation \mathbb{E}^i over $R^i \otimes_k R^i$, such that

- (i) there is a basic solution of the system \mathbb{E}^i , which forms a $R^i \otimes_k R^i$ -quasi-basis of \mathcal{K}_1^i ;
- (ii) the free variables in the basic solution form a $R^i \otimes_k R^i$ -quasi-basis of \mathcal{C}_1^i .

Proof Let $\Phi_{\underline{m}^0} = \sum_j v_j * V_j$, with $\underline{m}^0 = (1, \dots, 1)$, be the formal product Π of the pair $(\mathfrak{A}, \mathfrak{B})$, and let $\mathbb{E}^0 : \Phi_{\underline{m}^0} H^0 \equiv_{\prec(p,q)} H^0 \Phi_{\underline{m}^0}$ be a matrix equation, where the leading position of A_1 is (p, q) , $H^0 = 0$. Then the basic solution of \mathbb{E}^0 is $\{V_j\}_j$, the $R \otimes_k R$ -basis of \mathcal{K}_1 ; and the set of free variables is $\{v_j\}_j$, the $R \otimes_k R$ -basis of \mathcal{C}_1 . The assertion follows for $i = 0$.

Suppose a equation system \mathbb{E}^i for the pair $(\mathfrak{A}^i, \mathfrak{B}^i)$ satisfying (i) and (ii) has been obtained:

$$\mathbb{E}^i : \Phi_{\underline{m}^i} H^i \equiv_{\prec(p^i, q^i)} H^i \Phi_{\underline{m}^i}, \quad \Phi_{\underline{m}^i} = \sum_{X \in \mathcal{T}} \bar{w}_X^i * E_X + \sum_j \bar{v}_j^i * V_j, \quad (2.4-2)$$

where (p^i, q^i) is the leading position of A_1^i , $\bar{v}_j^i = (v_{j pq}^i)$ is the split of v_j of size $m_{s(v_j)}^i \times m_{t(v_j)}^i$, $\bar{w}_X^i = (w_{X pq}^i)$ is a square matrix of the size m_X^i , and $v_{j pq}^i, w_{X pq}^i$ are said to be *dotted elements* (not dotted arrows). Both of them are over $R^i \otimes_k R^i$, in fact, $v_{j pq}^i$ (resp. non-zero $w_{X pq}^i$) : $X^i \mapsto Y^i$, provided the entry of the identity matrix $E^i \in \mathcal{K}^i$ with the same row (resp. column) index of $v_{j pq}^i$ (resp. $w_{X pq}^i$) is $1_{X^i} \in R_{X^i}^i$ (resp. $1_{Y^i} \in R_{Y^i}^i$). We now construct the system \mathbb{E}^{i+1} .

(i) In the case of Regularization, we have $\underline{m}^{i+1} = \underline{m}^i$, then equation $\Phi_{\underline{m}^i} H^i \equiv_{(p^i, q^i)} H^i \Phi_{\underline{m}^i}$, combining with \mathbb{E}^i form the equation system $\mathbb{E}^{i+1} : \Phi_{\underline{m}^{i+1}} H^{i+1} \equiv_{\prec(p^{i+1}, q^{i+1})} H^{i+1} \Phi_{\underline{m}^{i+1}}$.

(ii) In the cases of Loop or Edge reductions of Lemma 2.3.2, set the first arrow $a_1^i : X^i \mapsto Y^i$ in \mathfrak{B}^i , denote by $\underline{n} = \vartheta^{i, i+1}(1, 1 \dots, 1)$ the size vector of $H^{i+1}(k)$ over \mathcal{T}^i , thus the size vector $\vartheta^{0i}(\underline{n}) = \underline{m}^{i+1}$ over \mathcal{T} according to Convention 2.3.1. Suppose $\bar{v}_j^i = (v_{j pq}^i)$, define a $n_{s(v_{j pq})} \times n_{t(v_{j pq})}$ -matrix block according to Theorem 2.1.3 (ii):

$$\bar{v}_j^{i+1} = \sum_{\alpha, \beta} (\mathbf{f}_{(s(v_{j pq}), \alpha)} \otimes_k \mathbf{e}_{(t(v_{j pq}), \beta)}) * v_{j pq}.$$

If $\bar{w}_X^i = (w_{X pq}^i)$, let \bar{w}_X^{i+1} be given below for loop and edge reduction respectively:

$$\begin{aligned} \bar{w}_X^{i+1} &= \sum_{\alpha, \beta} (\mathbf{f}_{(s(w_{X pq}), \alpha)} \otimes_k \mathbf{e}_{(t(w_{X pq}), \beta)}) * w_{X pq}^i \\ &+ \left\{ \begin{array}{l} (\sum_{h, j, l} (\mathbf{e}_{(h j 1)} \otimes_R \mathbf{f}_{(h j' l)}) * F_{h j j' l}) * E_{X^i}^i \mid \mathbf{p} \in X; \\ ((\mathbf{e}_{Z_2(X^i, 1)} \otimes_R \mathbf{f}_{Z_1(X^i, 2)}) * F_{Z_2 Z_1}) * E_{X^i}^i + ((\mathbf{e}_{Z_3(Y^i, 1)} \otimes_R \mathbf{f}_{Z_2(Y^i, 2)}) * F_{Z_3 Z_2}) * E_{Y^i}^i \mid \mathbf{p} \in X, \end{array} \right. \end{aligned}$$

where “ $\mid \mathbf{p} \in X$ ” means that the matrix blocks are restricted inside the (\mathbf{p}, \mathbf{p}) -th block for any $\mathbf{p} \in X$ partitioned by \mathcal{T} . Since the admissible bi-module depends only on the vertex set \mathcal{T} , $* \mid \mathbf{p} \in X$ are all equal when \mathbf{p} runs over X . Thus $\mathbb{E}^{i+1} : \Phi_{\underline{m}^{i+1}} H^{i+1} \equiv_{\prec(p^{i+1}, q^{i+1})} H^{i+1} \Phi_{\underline{m}^{i+1}}$ with $\Phi_{\underline{m}^{i+1}} = \sum_{X \in \mathcal{T}} \bar{w}_X^{i+1} * E_X + \sum_j \bar{v}_j^{i+1} * V_j$ in both cases, the theorem follows by induction.

$\Phi_{\underline{m}^i}$ and \mathbb{E}^i in Formula (2.4-2) are called the *variable matrix* of size vector \underline{m}^i , and the *defining system of the pair* $(\mathfrak{A}^i, \mathfrak{B}^i)$ respectively. It is clear that the system \mathbb{E}^i consisting of the equations at the $(\mathbf{p}_i, \mathbf{q}_i)$ -th block for some $1 \leq i \leq n$, where $(\mathbf{p}_i, \mathbf{q}_i)$ is the index of the leading block determined by A_i partitioned under \mathcal{T} .

Next, we give a deformed system based on the defining system. Fix some $0 < r < s$, write:

$$H^i = H_1^i + H_2^i, \quad H_1^i = \sum_{j=1}^{r-1} G_i^{j+1} * A_1^j, \quad H_2^i = \sum_{j=r}^{i-1} G_i^{j+1} * A_1^j.$$

Suppose $\mathfrak{A}^r = (R^r, \mathcal{K}^r, \mathcal{M}^r, H^r)$, the size vector $\underline{m}^{ri} = (m_1^{ri}, \dots, m_{t^r}^{ri}) = \vartheta^{ri}(1, \dots, 1)$ over \mathcal{T}^r . Take a trivially normalized quasi-basis $\{V_1^r, \dots, V_{m^r}^r\}$ of \mathcal{K}_1^r , define a variable matrix of size vector \underline{m}^{ri} over \mathcal{T}^r and a matrix equation:

$$\begin{aligned} \Psi_{\underline{m}^{ri}} &= \sum_{X^r \in \mathcal{T}^r} \bar{w}_{X^r}^{ri} * E_{X^r} + \sum_j \bar{v}_j^{ri} * V_j^r, \\ \Psi_{\underline{m}^{ri}}(H_1^i + H_2^i) &\equiv_{\prec(p^i, q^i)} (H_1^i + H_2^i) \Psi_{\underline{m}^{ri}}, \end{aligned}$$

where $\bar{v}_j^{ri} = (v_{jpq}^{ri})$ is the split of v_j^r of size $m_{s(v_j^r)}^{ri} \times m_{t(v_j^r)}^{ri}$, $\bar{w}_{X^r}^{ri} = (w_{X^r pq}^{ri})$ is a square matrix of the size $m_{X^r}^{ri}$. Furthermore, the equation system is equivalent to:

$$\begin{aligned} \mathbb{F}^{ri} : \Psi_{\underline{m}^{ri}} H_2^i &\equiv_{\prec(p^i, q^i)} \Psi_{\underline{m}^{ri}}^0 + H_2^i \Psi_{\underline{m}^{ri}}, \\ \Psi_{\underline{m}^{ri}}^0 &= H_1^i \Psi_{\underline{m}^{ri}} - \Psi_{\underline{m}^{ri}} H_1^i. \end{aligned} \quad (2.4-3)$$

Corollary 2.4.2 The matrix equations of formulae (2.4-2) and (2.4-3) are equivalent.

Now we give an altered Theorem used in section 5.2-5.4, which is very easy to be proved.

Theorem 2.4.3 (i) For each $0 \leq i \leq s$ in the sequence (2.4-1), there is a *defining system*:

$$\mathbb{E}^i : \Phi_{\underline{m}^i} H^i(k) \equiv_{\prec(p^i, q^i)} H^i(k) \Phi_{\underline{m}^i}; \quad \Phi_{\underline{m}^i} = \sum_{X \in \mathcal{T}} Z_X * E_X + \sum_j Z_j * V_j, \quad (2.4-4)$$

where $Z_X = (z_{pq}^X)_{m_X \times m_X}$, $\forall X \in \mathcal{T}$, and $Z_j = (z_{rs}^j)_{m_{s(v_j)} \times m_{t(v_j)}}$ for all quasi-basis matrix V_j of \mathcal{K}_1 in \mathfrak{A} , where z_{pq}^X, z_{rs}^j are algebraically independent variables over k . $\Phi_{\underline{m}^i}$ is called a *variable matrix*. Then the solution space of \mathbb{E}^i is $\mathcal{K}_0^i \oplus \mathcal{K}_1^i$ forgotten the $(R^i \oplus (R^i \otimes_k R^i))$ -structure.

(ii) Fix some $1 < r < s$, define algebraically independent variable matrices Z_Y^{ri} of size $m_Y^{ri}, \forall Y^r \in \mathcal{T}^r$, and Z_j^{ri} of size $m_{s(v_j^r)}^{ri} \times m_{t(v_j^r)}^{ri}, 1 \leq j \leq m^r$. Then the equation \mathbb{F}^{ri} below is equivalent to \mathbb{E}^i :

$$\begin{aligned} \mathbb{F}^{ri} : \Psi_{\underline{m}^{ri}} H_2^i(k) &\equiv_{\prec(p^i, q^i)} \Psi_{\underline{m}^{ri}}^0 + H_2^i(k) \Psi_{\underline{m}^{ri}}, \\ \Psi_{\underline{m}^{ri}}^0 &= H_1^i(k) \Psi_{\underline{m}^{ri}} - \Psi_{\underline{m}^{ri}} H_1^i(k), \quad \Psi_{\underline{m}^{ri}} = \sum_{Y^r \in \mathcal{T}^r} Z_{Y^r}^{ri} * E_{Y^r} + \sum_{j=1}^{m^r} Z_j^{ri} * V_j^r. \end{aligned} \quad (2.4-5)$$

Corollary 2.4.4 $\delta(a_1^i) = 0$ in \mathfrak{B}^i if and only if the equation $\mathbb{E}_{\prec(p^i, q^i)}^i$ is a linear combination of the equations of $\mathbb{E}_{\prec(p^i, q^i)}^i$.

Finally, we perform reduction procedure for the matrix bi-module problem given in Example 1.4.5 in order to show some concrete calculation to end the sub-section.

Example 2.4.5 (i) Making an edge reduction for the first arrow $a : X \rightarrow Y$ by $a \mapsto G^1 = (1_Z)$, we obtain an induced local bi-module problem \mathfrak{A}^1 (resp. induced boc \mathfrak{B}^1), with $R^1 = k1_Z$; $H^1 = (1_Z) * A$.

(ii) Making a loop reduction for $b : Z \rightarrow Z$ by $b \mapsto G^2 = J_2(0)1_X$, we obtain an induced local pair $(\mathfrak{A}^2, \mathfrak{B}^2)$ with $R^2 = k1_X, H^2 = \begin{pmatrix} 1_X & 0 \\ 0 & 1_X \end{pmatrix} * A + \begin{pmatrix} 0 & 1_X \\ 0 & 0 \end{pmatrix} * B$. Whose formal equation consists of two matrix equations:

$$\begin{aligned} &\begin{pmatrix} e & v \\ 0 & e \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} + \begin{pmatrix} u_{11}^2 & u_{12}^2 \\ u_{21}^2 & u_{22}^2 \end{pmatrix} \begin{pmatrix} 0 & 1_X \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1_X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11}^2 & v_{12}^2 \\ v_{21}^2 & v_{22}^2 \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} e & v \\ 0 & e \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
& \begin{pmatrix} e & v \\ 0 & e \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} + \begin{pmatrix} u_{11}^1 & u_{12}^1 \\ u_{21}^1 & u_{22}^1 \end{pmatrix} \begin{pmatrix} 0 & 1_X \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} u_{11}^2 & u_{12}^2 \\ u_{21}^2 & u_{22}^2 \end{pmatrix} \begin{pmatrix} 1_X & 0 \\ 0 & 1_X \end{pmatrix} \\
&= \begin{pmatrix} 1_X & 0 \\ 0 & 1_X \end{pmatrix} \begin{pmatrix} v_{11}^2 & v_{12}^2 \\ v_{21}^2 & v_{22}^2 \end{pmatrix} + \begin{pmatrix} 0 & 1_X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11}^1 & v_{12}^1 \\ v_{21}^1 & v_{22}^1 \end{pmatrix} + \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} e & v \\ 0 & e \end{pmatrix}.
\end{aligned}$$

Where $(c_{pq})_{2 \times 2}, (d_{pq})_{2 \times 2}$ are splits from c, d respectively, $e = e_X; v \in \mathcal{C}_1^2$ dual to $V = \begin{pmatrix} 0 & 1_X \otimes_k 1_X \\ 0 & 0 \end{pmatrix} * E_X \in \mathcal{K}_1^2, E_X = (1_X I_{10}, 1_X I_{10})$.

(iii) Making a loop mutation $c_{21} \mapsto (x)$, then 3 steps of regularization, such that $c_{22} \mapsto \emptyset, u_{21}^2 = xv; c_{11} \mapsto \emptyset, v_{21}^2 = vx; c_{12} \mapsto \emptyset, u_{11}^2 = v_{22}^2$, we obtain an induced pair $(\mathfrak{A}^3, \mathfrak{B}^3)$ with the differentials in \mathfrak{B}^3 :

$$\begin{cases} \delta(d_{21}) = xv - vx \\ \delta(d_{22}) = u_{21}^1 + u_{22}^2 - v_{22}^2 - d_{21}v \\ \delta(d_{11}) = u_{11}^2 - v_{11}^2 - v_{21}^1 + vd_{21} \\ \delta(d_{12}) = u_{11}^1 + u_{12}^2 - v_{12}^2 - v_{22}^1 - d_{11}v + vd_{22}. \end{cases}$$

(iv) Note that the blocks splitting from d_{22}, d_{11}, d_{12} will be going to \emptyset by regularization for any possible reductions for x and d_{21} .

3 Classification of Minimally Wild Bocses

The present section is devoted to classifying so called minimally wild bocses, which are divided into five classes. Then we prove the non-homogeneity for the bocses in four classes. But the last class has been proved to be strongly homogeneous.

3.1 Exact structures on representation categories of bocses

In this sub-section we will recall the exact structure on representation categories of bocses. We refer to [GR] and [DRSS] for the general concept of exact structure on additive categories with Krull-Schmidt property.

Let $\mathfrak{B} = (\Gamma, \Omega)$ be a bocs with a layer $L = (\Gamma'; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$. From now on we always assume that \mathfrak{B} is *triangular on the dotted arrows*, i.e. $\delta(v_j)$ involves only v_1, \dots, v_{j-1} .

The bocs $\mathfrak{B}_0 = (\Gamma, \Gamma)$ is called a *principal bocs* of \mathfrak{B} . The representation category $R(\mathfrak{B}_0)$ is just the module category $\Gamma\text{-mod}$.

Lemma 3.1.1 [O] Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered bocs, which is triangular on the dotted arrows, and has a principle bocs \mathfrak{B}_0 .

(i) If $\iota : M \rightarrow E$ is a morphism of $R(\mathfrak{B})$ with ι_0 injective, then there exists an isomorphism η and a commutative diagram in $R(\mathfrak{B})$, such that the bottom row is exact in $R(\mathfrak{B}_0)$. Dually, if $\pi : E \rightarrow N$ is a morphism of $R(\mathfrak{B})$ with π_0 surjective, then there exists an isomorphism η and a commutative diagram in $R(\mathfrak{B})$, such that the bottom row is exact in $R(\mathfrak{B}_0)$.

$$\begin{array}{ccccc}
M & \xrightarrow{\iota} & E & & E & \xrightarrow{\pi} & N \\
id \downarrow & & \downarrow \eta & & \eta \downarrow & & \downarrow id \\
0 & \longrightarrow & M & \xrightarrow{\iota'} & E' & \xrightarrow{\pi'} & N \longrightarrow 0
\end{array}$$

(ii) If $(e) : M \xrightarrow{\iota} E \xrightarrow{\pi} N$ with $\iota\pi = 0$ is a pair of composable morphisms in $R(\mathfrak{B})$ and $(e_0) : 0 \longrightarrow M \xrightarrow{\iota_0} E \xrightarrow{\pi_0} L \longrightarrow 0$ is exact in the category of vector spaces, then there exists an

isomorphism η and a commutative diagram in $R(\mathfrak{B})$:

$$(e) \quad \begin{array}{ccccc} M & \xrightarrow{\iota} & E & \xrightarrow{\pi} & N \\ id \downarrow & & \downarrow \eta & & \downarrow id \\ (e') \quad 0 & \longrightarrow & M & \xrightarrow{\iota'} & E' & \xrightarrow{\pi'} & N & \longrightarrow & 0 \end{array}$$

such that (e') is an exact sequence in $R(\mathfrak{B}_0)$. Moreover, by choosing a suitable basis of M, E', N , we are able to obtain $\iota'_X = (0, I)$ and $\pi'_X = (I, 0)^T$ for all $X \in \mathcal{T}$.

Lemma 3.1.2 Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered boc, which is triangular on the dotted arrows.

(i) $\iota : M \rightarrow E$ is monic in $R(\mathfrak{B})$ if $\iota_0 : M \rightarrow E$ is injective. Dually, $\pi : E \rightarrow N$ is epic in $R(\mathfrak{B})$ if $\pi_0 : E \rightarrow N$ is surjective.

(ii) A pair of composable morphisms $(e) : M \xrightarrow{\iota} E \xrightarrow{\pi} N$ with $\iota\pi = 0$ is exact in $R(\mathfrak{B})$, if $(e_0) : 0 \rightarrow M \xrightarrow{\iota_0} E \xrightarrow{\pi_0} N \rightarrow 0$ is exact as a sequence of vector spaces.

Proof. (i) If ι_0 is injective, Lemma 3.1.1 (i) gives a commutative diagram with $\iota' : M \rightarrow E'$ in $R(\mathfrak{B}_0)$. Given any morphism $\varphi : L \rightarrow M$ with $\varphi\iota = 0$, we have $\varphi\iota\eta = \varphi\iota' = 0$. Then $\varphi_0\iota'_0 = 0$ yields $\varphi_0 = 0$. And for any $v_l : X \rightarrow Y$, $\delta(v_l) = \sum_{i,j} u_i \otimes_\Gamma u_j$ with $u_i, u_j \in \oplus_{l' < l} \Gamma v_{l'} \Gamma$, using induction: $0 = (\varphi\iota')(v_l) = \varphi(v_l)\iota'_Y + \varphi_X \iota'(v_l) + \sum_{i,j} \varphi(u_i)\iota'(u_j) = \varphi(v_l)\iota'_Y$, which yields $\varphi(v_l) = 0$ by the injectivity of ι'_Y . Thus $\varphi = 0$ and ι is monic. The second one is proved dually.

(ii) We first prove that ι is the kernel of π . ① If (e_0) is exact, then (i) tells that ι is monic. ② $\iota\pi = 0$. ③ Lemma 3.1.1 (ii) gives a commutative diagram. If $\varphi : L \rightarrow E$ with $\varphi\pi = 0$, then $\varphi(\eta\pi') = 0$. Let $\xi = \varphi\eta$, then $\xi\pi' = 0$ implies that $\xi_X\pi'_X = 0$ and $\xi(v)\pi'_Y = 0$ for any vertex X and any dotted arrow $v : X \rightarrow Y$. Let $\varphi' : L \rightarrow M$ be given by $\varphi'_X\iota'_X = \xi_X$, $\varphi'(v)\iota'_Y = \xi(v)$, then we obtain $\varphi'\iota' = \xi$. Thus $\varphi'\iota'\eta^{-1} = \varphi$, i.e. $\varphi'\iota = \varphi$. Therefore ι is a kernel of π . It can be proved dually that π is a cokernel of ι . The proof is finished.

Let a layered boc $\mathfrak{B} = (\Gamma, \Omega)$ be triangular on the dotted arrows. We define a class \mathcal{E} of composable morphisms in $R(\mathfrak{B})$, such that $M \xrightarrow{\iota} E \xrightarrow{\pi} L$ in \mathcal{E} , provided that $\iota\pi = 0$ and

$$0 \longrightarrow M \xrightarrow{\iota_0} E \xrightarrow{\pi_0} L \longrightarrow 0 \quad (3.1-1)$$

is exact as a sequence of vector spaces. It is clear that \mathcal{E} is closed under isomorphisms.

Proposition 3.1.3 (Theorem 4.4.1 of [O] or [BBP]) Suppose a layered boc \mathfrak{B} is triangular on the dotted arrows, then \mathcal{E} defined by Formula (3.1-1) is an exact structure on $R(\mathfrak{B})$, and $(R(\mathfrak{B}), \mathcal{E})$ is an exact category.

In particular, the boc \mathfrak{B} associated to a matrix bi-module problem $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ is triangular. In fact, the basis $\mathcal{V} = \{V_1, \dots, V_m\}$ of \mathcal{K}_1 possesses a natural partial ordering: $V_i \prec V_j$ provided their leading position $(p_i, q_i) \prec (p_j, q_j)$. Since $\mathcal{K}_1 \subset \mathbb{N}_t(R \otimes_k R)$, $V_i \otimes_R V_j \in \sum_{V_l \prec V_i, V_j} R^{\otimes 3} \otimes_{R^{\otimes 2}} V_l$. Thus $\delta(v_l)$ contains only $v_i \otimes_R v_j$ with $v_i, v_j \prec v_l$.

Corollary 3.1.4 ([B1] and Lemma 7.1.1 of [O]) Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered boc.

(i) For any $M \in R(\mathfrak{B})$ with $\dim M = \underline{m}$. If $m_X \neq 0$ for some vertex $X \in \mathcal{T}_1$, then M is neither projective nor injective.

(ii) For any positive integer n , there are only finitely many iso-classes of indecomposable projectives and injectives in $R(\mathfrak{B})$ of dimension at most n .

Remark 3.1.5 ([BCLZ], Def. 4.4.1 of [O]) Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered boc, such that $(R(\mathfrak{B}), \mathcal{E})$ is an exact category. The almost split conflations has been defined in a general exact category, consequently in $R(\mathfrak{B})$.

(i) An indecomposable representation $M \in R(\mathfrak{B})$ is said to be *homogeneous* if there is an almost split conflation $M \xrightarrow{\iota} E \xrightarrow{\pi} M$.

(ii) The category $R(\mathfrak{B})$ (or boc \mathfrak{B}) is said to be *homogeneous* if for each positive integer n , almost all (except finitely many) iso-classes of indecomposable representations in $R(\mathfrak{B})$ with size at most n are homogeneous.

If \mathfrak{B} is of representation tame type, then $R(\mathfrak{B})$ is homogeneous [CB1].

(iii) The category $R(\mathfrak{B})$ (or boc \mathfrak{B}) is said to be *strongly homogeneous* if there exists neither projectives nor injectives, and all indecomposable representations in $R(\mathfrak{B})$ are homogeneous.

If \mathfrak{B} is a local boc with a layer $(R; \omega; a; v)$, $R = k[x, \phi(x)^{-1}]$, and the differential $\delta(a) = xv - vx$. Then $R(\mathfrak{B})$ is strongly homogeneous and representation wild type. In particular the induced boc given in Example 2.4.5 (iv) is strongly homogeneous [BCLZ].

Note that $(R(\mathfrak{B}), \mathcal{E})$ may not have any almost split conflation. For example, set quiver $Q = a \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} b$ and $\Gamma = kQ$. Then $R(\mathfrak{B})$ for the principal boc $\mathfrak{B} = (\Gamma, \Gamma)$ has no almost split conflations, see [V, ZL] for detail.

Recall from [CB1], let $\mathfrak{B} = (\Gamma, \Omega)$ be a minimal boc. Then for any $X \in \mathcal{T}_1$ with $R_X = k[x, \phi_X(x)^{-1}]$, and for any $\lambda \in k$, $\phi_X(\lambda) \neq 0$, there is an almost split conflation:

$$S(X, 1, \lambda) \xrightarrow{(01)} S(X, 2, \lambda) \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} S(X, 1, \lambda) \quad \text{in } R(\mathfrak{B}), \quad (3.1-2)$$

where $S(X, 1, \lambda)$ (resp. $S(X, 2, \lambda)$) is given by $k \bigcirc_{J_1(\lambda)}$ (resp. $= k \bigcirc_{J_2(\lambda)}$) at X , and $\{0\}$ at other vertices.

3.2 Almost split conflations in reductions

In this subsection we always assume that $\mathfrak{B} = (\Gamma, \Omega)$ is a layered boc with triangular property on the dotted arrows, and $\mathfrak{B}' = (\Gamma', \Omega')$ is an induced boc given by one of 8 reductions of sections 2.1-2.2. We will study the almost split conflations during the reductions.

Lemma 3.2.1 [B1] Let N' be an indecomposable representation in $R(\mathfrak{B}')$. If N' is non-projective (resp. non-injective) in $R(\mathfrak{B}')$, then so is $\vartheta(N')$ in $R(\mathfrak{B})$.

Lemma 3.2.2 [B1] (i) If $\iota' : M' \rightarrow E'$ is a morphism in $R(\mathfrak{B}')$ with $\vartheta(\iota') : \vartheta(M') \rightarrow \vartheta(E')$ being a left minimal almost split inflation in $R(\mathfrak{B})$, then so is ι' . Dually if $\pi' : E' \rightarrow N'$ is a morphism in $R(\mathfrak{B}')$ with $\vartheta(\pi') : \vartheta(E') \rightarrow \vartheta(N')$ being a right minimal almost split deflation in $R(\mathfrak{B})$, then so is π' .

(ii) If $(e') : M' \xrightarrow{\iota'} E' \xrightarrow{\pi'} M'$ is a conflation in $R(\mathfrak{B}')$ with $\vartheta(e') : \vartheta(M') \xrightarrow{\vartheta(\iota')} \vartheta(E') \xrightarrow{\vartheta(\pi')} \vartheta(M')$ being an almost split conflation in $R(\mathfrak{B})$, then so is (e') .

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a matrix bi-module problem with R being trivial and \mathfrak{B} the corresponding boc. Suppose that

$$(\mathfrak{A}, \mathfrak{B}) = (\mathfrak{A}^0, \mathfrak{B}^0), \dots, (\mathfrak{A}^i, \mathfrak{B}^i), (\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1}), \dots, (\mathfrak{A}^s, \mathfrak{B}^s) \quad (3.2-1)$$

is a sequence of reductions, such that \mathfrak{A}^{i+1} is obtained from \mathfrak{A}^i in the sense of Lemma 2.3.2 for $i = 0, \dots, s-1$; the first arrow $a_1^s \in \mathfrak{B}^s$ is a loop at X_1^s and $\delta(a_1^s) = 0$.

Theorem 3.2.3 With the assumption above, taken any indecomposable $M^s \in R(\mathfrak{A}^s)$ of size vector \underline{m}^s with $m_{x_1^s}^s = 1$, $M^s(a_1^s) = (\lambda)$. Suppose $\vartheta^{0s}(M^s) = M$ is homogeneous with an almost split conflation $(e) : M \rightarrow E \rightarrow M$ in $R(\mathfrak{A})$. Then for $i = 0, 1, \dots, s$ there exists an almost split conflation $(e^i) : M^i \rightarrow E^i \rightarrow M^i$ in $R(\mathfrak{A}^i)$, such that $\vartheta^{0i}(e^i) \simeq (e)$.

Proof We stress that $M_X^s = k$, and according to Theorem 2.3.3 (iii), as k -matrices:

$$M^s = H_{\underline{m}^s}^s(k) + \sum_j M^s(a_j^s) * A_j^s, \quad H_{\underline{m}^s}^s(k) = \sum_{j=1}^{s-1} B^{j+1} * A_1^j.$$

The theorem is obviously true for $i = 0$. Suppose the assertion is valid for some $0 \leq i < s$, we will reach the $(i + 1)$ -th stage. Set $\underline{m}^i = \vartheta^{is}(\underline{m}^s)$, Formula (3.2-2) gives as k -matrices:

$$M^i = M^s = H_{\underline{m}^i}^i(k) + B^{i+1} * A_1^i + \sum_{j=2}^{n^i} M^i(a_j^i) * A_j^i. \quad (3.2-2)$$

If there exists an object $E^i = H_{\underline{m}^i}^i(k) + \sum_{j=1}^{m^i} E^i(a_j^i) * A_j^i \in R(\mathfrak{A}^i)$, and an almost split conflation $(e^i) : M^i \xrightarrow{\iota^i} E^i \xrightarrow{\pi^i} M^i$ in $R(\mathfrak{A}^i)$, such that $\vartheta^{0i}(e^i) \simeq (e)$, we will prove in (i) and (ii) below, that there exists some isomorphism $\eta : E^i \mapsto \hat{E}^i$ in $R(\mathfrak{A}^i)$ with $\hat{E}^i(a_1^i) = B^{i+1} \oplus B^{i+1}$.

If this is the case, let $a_1^i : X \rightarrow Y$, S_X and S_Y are invertible matrices determined by changing certain rows and columns of $B^{i+1} \oplus B^{i+1}$, such that $S_X^{-1}(B^{i+1} \oplus B^{i+1})S_Y = I_2 \otimes_k B^{i+1}$, define a matrix $S = \sum_{Z \in \mathcal{T}^i} P_Z * E_Z$ with $S_Z = I_{m_Z}$ for $Z \in \mathcal{T}^i \setminus \{X, Y\}$. Then as k -matrices:

$$\begin{aligned} R(\mathfrak{B}^i) \ni S^{-1} \hat{E} S &= H_{\underline{m}^i}^i(k) + (I_2 \otimes_k B^{i+1}) * A_1^i + \sum_{j=2}^{n^i} S_{s(a_j^i)}^{-1} \hat{E}(a_j^i) S_{t(a_j^i)} * A_j^i \\ &\stackrel{\text{define}}{=} H_{\underline{m}^{i+1}}^{i+1}(k) + \sum_{j=1}^{n^{i+1}} E^{i+1}(a_j^{i+1}) * A_j^{i+1} = E^{i+1} \in R(\mathfrak{A}^{i+1}). \end{aligned}$$

Thus we obtain a conflation $(\hat{e}^i) : M^i \rightarrow S \hat{E}^i S^{-1} \rightarrow M^i$ equivalent to (e^i) in $R(\mathfrak{A}^i)$, and an almost split conflation (e^{i+1}) in $R(\mathfrak{A}^{i+1})$ with $\vartheta^{i,i+1}(e^{i+1}) \simeq (\hat{e}^i) \simeq (e^i)$ by Lemma 3.2.2 (ii). Consequently $\vartheta^{0,i+1}(e^{i+1}) \simeq \vartheta^{0i}(e^i) \simeq (e)$.

(i) If $\delta(a_1^i) = v_1^i \neq 0$, after a regularization, Set $\hat{E}^i = \{\hat{E}_Z^i \mid \dim(E_Z^i) = 2m_Z^i, Z \in \mathcal{T}^i\}$ be a set of vector spaces. Let $\eta : E^i \rightarrow \hat{E}^i$, such that $\eta_Z = I_{2m_Z^i}$ for any $Z \in \mathcal{T}^i$, and $\eta(v_1^i) = E^i(a_1^i)$, $\eta(v_j) = 0$ for any $j = 2, \dots, m^i$. Let $\hat{E}^i = \eta^{-1} E^i \eta \in R(\mathfrak{A}^i)$, then $\hat{E}^i(a_1^i) = E^i(a_1^i) - \eta(v_1^i) = (0)_{2m_X \times 2m_Y} = B^{i+1} \oplus B^{i+1}$ as desired.

(ii) If $\delta(a_1^i) = 0$ in the case of edge or loop reduction, the proof is divided into three parts.

① We define an object $L^s = H_{\underline{m}^s}^s(k) + \sum_{i=1}^{n^s} L^s(a_i^s) \in R(\mathfrak{A}^s)$ with $L^s(a_1^s) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Let $\varphi^s : L^s \rightarrow M^s$ be a morphism in $R(\mathfrak{A}^s)$, such that $\varphi_{X^s}^s = \begin{pmatrix} I_{X^s} \\ 0 \end{pmatrix}, \forall X^s \in \mathcal{T}^s$, and $\varphi^s(v_j^s) = 0$ for any dotted arrow v_j^s in \mathfrak{B}^s . Clearly, φ^s is not a split epimorphism. Thus $\vartheta^{is}(\varphi^s) : \vartheta^{is}(L^s) \mapsto \vartheta^{is}(M^s) = M^i$ is not a split epimorphism, since the functor ϑ^{is} is fully faithful.

② Because $\vartheta^{is}(L^s) = H_{\underline{m}^i}^i(k) + (I_2 \otimes_k B^{i+1}) * A_1^i + \sum_{j=2}^{n^i} \vartheta^{is}(L^s)(a_i) * A_j^i$ by Formula (3.2-2), we are able to construct an object L^i with $L^i(a_1^i) = B^{i+1} \oplus B^{i+1}$ by changing certain rows and columns in $I_2 \otimes_k B^{i+1}$, then there is an isomorphism $L^i \xrightarrow{\eta} \vartheta^{is}(L^s)$. We obtain a lifting $\tilde{\varphi}^i$ of $\varphi^i = \vartheta^{is}(\varphi^s) \eta$ in $R(\mathfrak{A}^i)$, such that $\varphi^i = \tilde{\varphi}^i \pi^i$, since $\pi^i : E^i \rightarrow M^i$ is right almost split in $R(\mathfrak{A}^i)$. The triangle and the square below are both commutative:

$$\begin{array}{ccc} & L^i & \\ \tilde{\varphi}^i \swarrow & & \searrow \varphi^i \\ E^i & \xrightarrow{\pi^i} & M^i \end{array} \quad \begin{array}{ccc} L_X^i & \xrightarrow{L^i(a_1^i)} & L_Y^i \\ \tilde{\varphi}_X^i \downarrow & & \downarrow \tilde{\varphi}_Y^i \\ E_X^i & \xrightarrow{E^i(a_1^i)} & E_Y^i \end{array}$$

③ We may assume according to Lemma 3.1.1 that in the sequence (e^i) , $\iota_Z^i = (0 \ I_Z), \pi_Z^i = \begin{pmatrix} I_Z \\ 0 \end{pmatrix}, \forall Z \in \mathcal{T}^i$, then $E^i(a_j^i) = \begin{pmatrix} M^i(a_j^i) & K_j^i \\ 0 & M^i(a_j^i) \end{pmatrix}$. The commutative triangle forces $\tilde{\varphi}_Z^i = \begin{pmatrix} I_Z & C_Z \\ 0 & D_Z \end{pmatrix}$ for each $Z \in \mathcal{T}^i$. The commutative square yields an equality

$$\begin{pmatrix} I_X & C_X \\ & D_X \end{pmatrix} \begin{pmatrix} B^{i+1} & K_1^i \\ & B^{i+1} \end{pmatrix} = \begin{pmatrix} B^{i+1} & \\ & B^{i+1} \end{pmatrix} \begin{pmatrix} I_Y & C_Y \\ & D_Y \end{pmatrix}.$$

Let $\hat{E}^i = \{\hat{E}_Z^i \mid \dim(E_Z^i) = 2m_Z^i, Z \in \mathcal{T}^i\}$ be a set of vector spaces. Define a set of isomorphisms $\eta : E^i \rightarrow \hat{E}^i$, such that $\eta_X = \begin{pmatrix} I_X & C_X \\ 0 & I_X \end{pmatrix}$, $\eta_Y = \begin{pmatrix} I_Y & C_Y \\ 0 & I_Y \end{pmatrix}$, and $\eta_Z = I_{2m_Z^i}$ for $Z \in \mathcal{T}^i \setminus \{X, Y\}$;

$\eta(v_j) = 0$ for any $j = 1, \dots, m^i$. Let $\hat{E}^i(a_j^i) = \eta_{s(a_j^i)} E^i(a_j^i) \eta_{t(a_j^i)}^{-1}$ for $j = 1, \dots, n^i$, we obtain an object $\hat{E}^i = \eta E \eta^{-1} \in R(\mathfrak{A}^i)$ with $\hat{E}^i(a_1^i) = B^{i+1} \oplus B^{i+1}$ as desired. The proof is completed.

Suppose \mathfrak{B} is a boc, and boc \mathfrak{B}' is induced from \mathfrak{B} by deletion of a vertex set $\mathcal{T}' \subset \mathcal{T}$. If $M' \in R(\mathfrak{B}')$, $M = \vartheta(M') \in R(\mathfrak{B})$ is homogeneous with an almost split conflation $(e) : M \rightarrow E \rightarrow M$, then there exists an almost split conflation $(e') \in R(\mathfrak{B}')$ with $\vartheta(e') \simeq (e)$. In fact, $\dim(E_X) = 2\dim(M_X)$ for any $X \in \mathcal{T}$ from the definition of the exact structure (3.1-1). So that $E_X \neq \{0\}$ if and only if $M_X \neq \{0\}$, and thus $X \notin \mathcal{T}'$, $E \in R(\mathfrak{B}')$.

Suppose we have the following sequence with the first part up to the s -th pair is given by Formula (3.2-1):

$$(\mathfrak{A}, \mathfrak{B}) = (\mathfrak{A}^0, \mathfrak{B}^0) \dots, (\mathfrak{A}^s, \mathfrak{B}^s), (\mathfrak{A}^{s+1}, \mathfrak{B}^{s+1}) \dots, (\mathfrak{A}^i, \mathfrak{B}^i), (\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1}), \dots, (\mathfrak{A}^t, \mathfrak{B}^t). \quad (3.2-3)$$

Assume that in the sequence (3.2-3), \mathfrak{B}^s is local with $\mathcal{T}^s = \{X\}$; \mathfrak{B}^{s+1} is induced from \mathfrak{B}^s by a loop mutation; and the reduction from \mathfrak{B}^i to \mathfrak{B}^{i+1} is given by a localization then a regularization, such that $R^{i+1} = k[x, \phi^{i+1}(x)^{-1}]$ for $s < i < t$, and \mathfrak{B}^t is minimal.

Corollary 3.2.4 Suppose $M^t \in R(\mathfrak{B}^t)$ with $M_X^t = k$, $M^t(x) = (\lambda)$ being regular, such that $\vartheta^{0t}(M^t) = M \in R(\mathfrak{B})$ is homogeneous with an almost split conflation (e) . Then there exists an almost split conflation (e^t) of Formula (3.1-2) in $R(\mathfrak{B}^t)$, such that $\vartheta^{0t}(e_\lambda^t) \simeq (e)$.

Proof Set $M^s = \vartheta^{st}(M^t)$, then $M^s(a_1^s) = (\lambda)$. Lemma 3.2.3 gives an almost split conflation (e^s) in $R(\mathfrak{B}^s)$ with $\vartheta^{0s}(e^s) \simeq (e)$. Moreover, $R(\mathfrak{A}^{s+1}) \simeq R(\mathfrak{A}^s)$, and for $i \geq s$, $R(\mathfrak{A}^{i+1})$ is equivalent to a subcategory of $R(\mathfrak{A}^i)$ consisting of the objects M^i with the eigenvalues of $M^i(x)$ are not the roots of $\phi^{i+1}(x)$. The proof is finished.

Assume that in the sequence (3.2-3), \mathfrak{B}^s has two vertices X, Y , the first arrow $a_1^s : X \mapsto X$ with $\delta(a_1^s) = 0$, \mathfrak{B}^{s+1} is induced from \mathfrak{B}^s by a loop mutation $a_1^s \mapsto (x)$; there exists a certain index $s < l < t$, such that $a_1^{s+1}, \dots, a_{l-s}^{s+1}$ are either loops at X , or edges from Y to X , especially $a_{l-s}^{s+1} : Y \mapsto X$. The reduction from \mathfrak{B}^i to \mathfrak{B}^{i+1} is given by

- ① a localization then a regularization for the first loop at X , or a regularization for the first edge; or a reduction given by proposition 2.2.6 for the first edge, when $s < i < l$;
- ② a reduction given by proposition 2.2.7, when $i = l$, then \mathfrak{B}^{l+1} is local;
- ③ a localization then a regularization for the first loop, when $l < i < t$.

Corollary 3.2.5 Suppose $M^t \in R(\mathfrak{B}^t)$ with $M_Z^t = k$, $M^t(z) = \lambda$ being regular, such that $\vartheta^{0t}(M^t) = M \in R(\mathfrak{B})$ is homogeneous with an almost split conflation (e) , then there exists an almost split conflation $(e_\lambda^t) \in R(\mathfrak{B}^t)$ given by Formula (3.1-2), such that $\vartheta^{0t}(e_\lambda^t) \simeq (e)$.

Proof Set $M^s = \vartheta^{st}(M^t)$, then $M_X^s = k$, $M_Y^s = k$, and $M^s(a_1^s) = (\lambda)$. Lemma 3.2.3 gives an almost split conflation $(e^s) : M^s \rightarrow E^s \rightarrow M^s$ in \mathfrak{B}^s . Since $R(\mathfrak{B}^{s+1}) \simeq R(\mathfrak{B}^s)$, we use induction starting from \mathfrak{B}^{s+1} . Suppose for some $i > s$, there exists an almost split conflation

$$(e^i) : M^i \xrightarrow{\iota^i} E^i \xrightarrow{\pi^i} M^i \in R(\mathfrak{B}^i) \quad \text{with} \quad M^i = \vartheta^{it}(M^t), \quad \vartheta^{si}(e^i) \simeq (e^s),$$

we now construct an almost split conflation (e^{i+1}) with $\vartheta^{i,i+1}(e^{i+1}) \simeq (e^i)$.

①, A regularization for an edge yields an equivalence; the proof of a regularization for a loop is similar to that of Corollary 3.2.4.

Denote by $a_1^i : Y \rightarrow X$ the first edge of \mathfrak{B}^i with $\delta(a_1^i) = 0$. By Lemma 3.1.1 (ii), we may assume $\iota^i = (0 \ 1), \pi^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Proposition 2.2.6 tells $M^i(a_1^i) = (0)$, therefore $E^i(a_1^i) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. If $0 \neq b \in k$, define $L \in R(\mathfrak{A}^i)$ with $L_X = k^2 = L_Y, L(x) = J_2(\lambda), L(a_j^i) = 0, \forall j$, and a morphism $g : L \rightarrow M^i$ with $g_X = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = g_Y, g(v) = 0$ for all dotted arrows. Then g is not a split epimorphism, there exists a lifting $\tilde{g} : L \rightarrow E^i$ with $\tilde{g}_X = \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix}, \tilde{g}_Y = \begin{pmatrix} 1 & c' \\ 0 & d' \end{pmatrix}$. Since

\tilde{g} is a morphism, $L^i(a_1^i)\tilde{g}_X = \tilde{g}_Y E^i(a_1^i)$, which yields $0 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ a contradiction. Therefore $b = 0$, $E^i(a_1^i) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Set $E^{i+1} \in R(\mathfrak{B}^{i+1})$ with $E^{i+1}(x) = J_2(\lambda)$, $E^{i+1}(a_{j-1}^{i+1}) = E^i(a_j^i)$ for $j = 2, \dots, n^i$, then $\vartheta^{i,i+1}(E^{i+1}) = E^i$.

② Proposition 2.2.7 tells $M^i(a_1^i) = (1)$, similar to proof ①, we may assume $E^i(a_1^i) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore there is a $E^{i+1} \in R(\mathfrak{B}^{i+1})$ with $E^{i+1}(a_{j-1}^{i+1}) = E^i(a_j^i)$ for $j = 2, \dots, n^i$, such that $\vartheta^{i,i+1}(E^{i+1}) \simeq E^i$.

③ Similar to the proof of Corollary 3.2.4, the proof is completed.

Lemma 3.2.6 (i) Suppose that $f(x, y) \in k[x, y]$ with $f(\lambda, \mu) \neq 0$. Let W_λ, W_μ be Weyr matrices of size m, n and eigenvalues λ, μ respectively, and $V = (v_{ij})_{m \times n}$ with v_{ij} being k -linearly independent. Let $f(W_\lambda, W_\mu)V = (u_{ij})_{m \times n}$. Then u_{ij} are also k -linearly independent for $1 \leq i \leq m, 1 \leq j \leq n$.

(ii) Let \mathfrak{B} be a bocs with $R = R_X \times R_Y$, where $R_X = k[x, \phi_X(x)^{-1}]$, $R_Y = k[y, \phi_Y(y)^{-1}]$, and $a_i : X \mapsto Y$. Define $\delta^0(a_i)$ to be a part of $\delta(a_i)$ without the terms involving $a_j, \forall j < i$. It is possible that $X = Y$, in this case x stands for the multiplication from left and y from right.

$$\begin{cases} \delta^0(a_1) = f_{11}(x, y)v_1 \\ \delta^0(a_2) = f_{21}(x, y)v_1 + f_{22}(x, y)v_2, \\ \dots \dots \\ \delta^0(a_n) = f_{n1}(x, y)v_1 + f_{n2}(x, y)v_2 + \dots f_{nn}(x, y)v_n, \end{cases}$$

where $f_{ii}(x, y) \in R_X \times R_Y$ are invertible for $i = 1, 2, \dots, n$. Set $x \mapsto W_X$ of size m with eigenvalues λ satisfying $\phi_X(\lambda) \neq 0$, $y \mapsto W_Y$ of size n with eigenvalues μ satisfying $\phi_Y(\mu) \neq 0$. Then the solid arrows splitting from a_1, \dots, a_n are all going to \emptyset by regularization in further reductions.

Proof (i) Set $f(x, y) = \sum_{i,j \geq 0} \alpha_{ij} x^i y^j$, then $f(W_\lambda, W_\mu)V = \sum_{i,j \geq 0} \alpha_{ij} W_\lambda^i V W_\mu^j$, we have $u_{ij} = f(\lambda, \mu)v_{ij} + \sum_{(i',j') \succ (i,j)} f_{i'j'}(\lambda, \mu)v_{i'j'}$ with $f_{i'j'}(x, y) \in k[x, y]$. The conclusion follows by induction on the ordered index set $\{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. (ii) follows by (i) inductively.

3.3 Minimal wild bocses

In this sub-section we will define five classes of minimal wild bocses in order to prove the main theorem. Our classification releases on the well-known Drozd's wild configurations, which is a refinement at some last reduction steps of those.

Proposition 3.3.1 ([CB1], Prop.3.10) Let $\mathfrak{B} = (\Gamma, \Omega)$ be a bocs of representation wild type with a layer $L = (R; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$. We are bound to meet one of the following configurations at some stage of reductions:

Case 1 $X \in \mathcal{T}_1$, $Y \in \mathcal{T}_0$ (or dual), $\delta(a_1) = 0$.

Case 2 $X, Y \in \mathcal{T}_1$ (possibly $X = Y$), $\delta(a_1) = f(x, y)v_1$ with $f(x, y) \in k[x, y, (\phi_X(x)\phi_Y(y))^{-1}]$ non-invertible.

We first fix some notations. Let \mathfrak{B} be a bocs with dotted arrows v_1, \dots, v_m . Consider a vector space \mathcal{S} over $k(x, y, z)$, the fractional field of the polynomial ring $k[x, y, z]$ of three indeterminates. Suppose there is a linear combination:

$$G = f_1(x, y, z)v_1 + \dots + f_m(x, y, z)v_m, \quad f_i(x, y, z) \in k[x, y, z]. \quad (3.3-1)$$

Let $h(x, y, z)$ be the greatest common factor of f_1, \dots, f_m , then $f_1/h, \dots, f_m/h$ are co-prime. There exists some $s_i(x, y, z) \in k[x, y, z]$ for $i = 1, \dots, m$, such that $\sum_{i=1}^m s_i(g_i/h) = c(x, y) \in k[x, y]$. Then $G = h \sum_{i=1}^m (g_i/h)v_j$. Since $S = k[x, y, z, c(x, y)^{-1}]$ is a Hermite ring, there exists

some invertible $F(x, y; z) \in \mathbb{M}_m(S)$ with the first column $(f_1/h, \dots, f_m/h)$. Then we make a base change of the form $(w_1, \dots, w_m) = (v_1, \dots, v_m)F$, $G = h(x, y; z)w_1$.

Let $f(x, \bar{x}) \in k[x, \bar{x}]$, in the form $f(x, \bar{x})v$, x stands for the left multiple and \bar{x} for the right.

Classification 3.3.2 Let a wild boc \mathfrak{B}^0 be given by Proposition 3.3.1. Then we are bound to meet a boc \mathfrak{B} with a layer $L = (R; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$ in one of the five classes at some stage of reductions, which are called *minimally wild bocses*.

Suppose the boc \mathfrak{B} has two vertices $\mathcal{T} = \{X, Y\}$, such that the induced local boc \mathfrak{B}_X is tame infinite with $R_X = k[x, \phi_X(x)^{-1}]$.

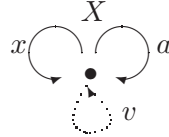
MW1 \mathfrak{B}_Y is finite with $R_Y = k1_Y$, $\delta(a_1) = 0$:

$$x \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} X \xrightarrow{a_1} Y.$$

MW2 \mathfrak{B}_Y is tame infinite with $R_Y = k[y, \phi_Y(y)^{-1}]$, $\delta(a_1) = f(x, y)v_1$, such that $f(x, y) \in k[x, y, \phi_X(x)^{-1}\phi_Y(y)^{-1}]$ is non-invertible:

$$x \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} X \xrightarrow{a_1} Y \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} y.$$

Suppose now we have a local boc \mathfrak{B} with $R = k[x, \phi(x)^{-1}]$:



MW3 The differential δ^0 of the solid arrows of \mathfrak{B} are:

$$\begin{cases} \delta^0(a_1) &= f_{11}(x, \bar{x})w_1, \\ \dots & \dots \\ \delta^0(a_n) &= f_{n1}(x, \bar{x})w_1 + \dots + f_{nn}(x, \bar{x})w_n, \end{cases} \quad (3.3-2)$$

where w_1, \dots, w_n are given by base changes, $f_{ii}(x, x) \in k[x, \phi(x)^{-1}]$ are all invertible; $f_{11}(x, \bar{x}) \in k[x, \bar{x}, \phi(x)^{-1}\phi(\bar{x})^{-1}]$ is non-invertible.

Now suppose there exists some $1 \leq n_1 \leq n$, such that:

$$\begin{cases} \delta^0(a_1) &= f_{11}(x, \bar{x})w_1, \\ \dots & \dots \\ \delta^0(a_{n_1-1}) &= f_{n_1-1,1}(x, \bar{x})w_1 + \dots + f_{n_1-1,1}(x, \bar{x})w_{n_1-1}, \\ \delta^0(a_{n_1}) &= f_{n_1,1}(x, \bar{x})w_1 + \dots + f_{n_1,n_1-1}(x, \bar{x})w_{n_1-1} + f_{n_1,n_1}(x, \bar{x})\bar{w}, \end{cases} \quad (3.3-3)$$

where $f_{ii}(x, x) \in k[x, \phi(x)^{-1}]$, $1 \leq i < n_1$, are invertible; $\bar{w} = 0$, or $\bar{w} \neq 0$ but $f_{n_1,n_1}(x, x) = 0$. Denote by x_1 the solid arrow a_{n_1} , there exist a polynomial $\psi(x, x_1)$ being divided by $\phi(x)$. Write δ^1 the part of differential δ by deleting all the terms involving some solid arrow except x, x_1 , and the further unraveling on x is restricted to $x \mapsto (\lambda)$ for $\psi(\lambda, x_1) \neq 0$, such that

$$\begin{cases} \delta^1(a_{n_1+1}) &= K_{n_1+1} + f_{n_1+1,n_1+1}(x, x_1, \bar{x}_1)w_{n_1+1}, \\ \dots & \dots \\ \delta^1(a_n) &= K_n + f_{n,n_1+1}(x, x_1, \bar{x}_1)w_{n_1+1} + \dots + f_{nn}(x, x_1, \bar{x}_1)w_n, \end{cases} \quad (3.3-4)$$

with $K_i = \sum_{j=1}^{n_1-1} f_{ij}(x, x_1, \bar{x}_1)w_j$, where w_{n_1+1}, \dots, w_n are given by the base changes (3.3-1) step by step, $f_{ii}(x, x_1, \bar{x}_1) \in k[x, x_1, \bar{x}_1, \phi(x, x_1)^{-1}\phi(x, \bar{x}_1)^{-1}]$ are all invertible for $n_1 < i \leq n$.

MW4 $\bar{w} = 0$, or $\bar{w} \neq 0$ and $(x - \bar{x})^2 \mid f_{n_1 n_1}(x, \bar{x})$ in Formula (3.3-2).

MW5 $\bar{w} \neq 0$ and $(x - \bar{x})^2 \nmid f_{n_1 n_1}(x, \bar{x})$ in Formula (3.3-2).

The proof of the classification depends on the classification of local bocses, whose proof is based on Formulae (3.3-2)-(3.3-9) and two Lemmas below.

Let \mathfrak{B} be a local boc having a layer $L = (R; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$. If $R = k1_X$ is trivial, then the differentials of the solid arrows have two possibilities. First:

$$\begin{cases} \delta^0(a_1) &= f_{11}w_1, \\ \dots &\dots \\ \delta^0(a_n) &= f_{n1}w_1 + \dots + f_{nn}w_n, \end{cases} \quad (3.3-5)$$

with $f_{ij} \in k, h_{ii} \neq 0$ for $1 \leq i \leq n$, and w_1, \dots, w_n given by base changes. Second, there exists some $1 \leq n_0 \leq n$ such that:

$$\begin{cases} \delta^0(a_1) &= f_{11}w_1, \\ \dots &\dots \\ \delta^0(a_{n_0-1}) &= f_{n_0-1,1}w_1 + \dots + f_{n_0-1,n_0-1}w_{n_0-1}, \\ \delta^0(a_{n_0}) &= f_{n_0,1}w_1 + \dots + f_{n_0,n_0-1}w_{n_0-1}, \end{cases} \quad (3.3-6)$$

with $f_{ij} \in k, f_{ii} \neq 0$ for $1 \leq i < n_0$. Set $a_i \mapsto \emptyset, i = 1, \dots, n_0 - 1$ by a series of regularization, then $a_{n_0} \mapsto (x)$ by a loop mutation, we obtain an induced local boc \mathfrak{B}' .

Without loss of generality, we may still denote \mathfrak{B}' by \mathfrak{B} with the layer L , but $R = k[x]$. The differentials δ^0 have again two possibilities. First one is given by Formula (3.3-2), such that $f_{ii}(x, x) \neq 0$ for $i = 1, \dots, n$. Define a polynomial:

$$\phi(x) = \prod_{i=1}^n c_i(x) f_{ii}(x, x) \quad (3.3-7)$$

with $c_i(x)$ appearing at the localization in order to do a base change before the i -th step of regularization, thus $f_{ii}(x, x)$ are invertible in $k[x, \phi(x)^{-1}]$.

Lemma 3.3.3 Let \mathfrak{B} be a boc given by Formula (3.3-2) with a polynomial $\phi(x)$ (3.3-7). There exist two cases:

- (i) $f_{ii}(x, \bar{x}) \in k[x, \bar{x}, \phi(x)^{-1}\phi(\bar{x})^{-1}]$ are all invertible for $1 \leq i \leq n$;
- (ii) There exists some minimal $1 \leq s \leq n$, such that $f_{ss}(x, \bar{x}) \in k[x, \bar{x}, \phi(x)^{-1}\phi(\bar{x})^{-1}]$ is non-invertible.

The second possibility of the differential δ^0 in the case $R = k[x]$ is given by Formula (3.3-3) for some fixed $1 \leq n_1 \leq n$, where $f_{ii}(x, x) \neq 0$ for $1 \leq i < n_1$; $\bar{w} = 0$, or $\bar{w} \neq 0$ but $f_{n_1, n_1}(x, x) = 0$. Define

$$\phi(x) = \begin{cases} \prod_{i=1}^{n_1-1} c_i(x) f_{ii}(x, x), & \bar{w} = 0; \\ c_{n_1}(x) \prod_{i=1}^{n_1-1} c_i(x) f_{ii}(x, x), & \bar{w} \neq 0, \end{cases} \quad (3.3-8)$$

then $f_{ii}(x, x), 1 \leq i < n_1$, are invertible in $k[x, \phi(x)^{-1}]$.

There are two possibilities in the further reductions for the third time. First possibility is given by Formula (3.3-4), such that $f_{ii}(x, x_1, x_1) \neq 0$ for $n_1 < i \leq n$. There is a sequence of localizations given by polynomials $c_i(x, x_1)$ in order to do base changes before each regularizations for $i = n_1 + 1, \dots, n$. Define a polynomial

$$\psi(x, x_1) = \phi(x) \prod_{i=n_1+1}^n c_i(x, x_1) f_{ii}(x, x_1, x_1). \quad (3.3-9)$$

Lemma 3.3.4 Let the boc \mathfrak{B} be given by Formulae (3.3-3)-(3.3-4) with polynomials $\phi(x)$ in (3.3-8), and $\psi(x, x_1)$ in (3.3-9). We obtain two cases.

- (i) There exists some $\lambda \in k$ with $\psi(\lambda, x_1) \neq 0$, and a minimal $n_1 + 1 \leq s \leq n$, such that $f_{ss}(\lambda, x_1, \bar{x}_1) \in k[x_1, \bar{x}_1, \psi(\lambda, x_1)^{-1}\psi(\lambda, \bar{x}_1)^{-1}]$ being non-invertible, i.e., after making a

unraveling $x \mapsto (\lambda)$, and then a series of regularization $a_i \mapsto \emptyset, w_i = 0$ for $i = 1, \dots, n_1 - 1$, the induced local boc $\mathfrak{B}_{(\lambda)}$ with $R_{(\lambda)} = k[x_1, \psi(\lambda, x_1)^{-1}]$ satisfies Lemma 3.3.3 (ii).

(ii) For any $\lambda \in k$ with $\psi(\lambda, x_1) \neq 0$, $f_{ii}(\lambda, x_1, \bar{x}_1) \in k[x_1, \bar{x}_1, \psi(\lambda, x_1)^{-1}\psi(\lambda, \bar{x}_1)^{-1}]$ are invertible for $n_1 < i \leq n$, i.e., the induced boc $\mathfrak{B}_{(\lambda)}$ with $R_{(\lambda)} = k[x_1, \psi(\lambda, x_1)^{-1}]$ has 3.3.3 (i).

(iii) The case (ii) is equivalent to $f_{ii}(x, x_1, \bar{x}_1) \in k[x, x_1, \bar{x}_1, \phi(x, x_1)^{-1}\phi(x, \bar{x}_1)^{-1}]$ being invertible for all $n_1 < i \leq n$.

Proof (ii) \implies (iii) If there exists some $n_1 < s \leq n$ with $f_{ss}(x, x_1, \bar{x}_1)$ non-invertible, then f_{ss} contains a non-trivial factor $g(x, x_1, \bar{x}_1)$ co-prime to $\psi(x, x_1)\psi(x, \bar{x}_1)$. Consider the variety $V = \{(\alpha, \beta, \gamma) \in k^3 \mid g(\alpha, \beta, \gamma) = 0, \psi(\alpha, \beta)\phi(\alpha, \gamma) = 0\}$. Since $\dim(V) \leq 1$, there exists a co-finite subset $\mathcal{L} \subset k$, such that $\forall \lambda \in \mathcal{L}$, the plane $x = \lambda$ of k^3 intersects V at only finitely many points. Thus $g(\lambda, x, \bar{x}_1)$ and $\psi(\lambda, x_1)\psi(\lambda, \bar{x}_1)$ are co-primes, consequently $g(\lambda, x, \bar{x}_1)$, thus $f_{ss}(\lambda, x_1, \bar{x}_1) \in k[x_1, \bar{x}_1, \psi(\lambda, x_1)^{-1}\psi(\lambda, \bar{x}_1)^{-1}]$ is not invertible.

(iii) \implies (ii) If $f_{ii}(x, x_1, \bar{x}_1) \in k[x, x_1, \bar{x}_1, \psi(x, x_1)^{-1}\psi(x, \bar{x}_1)^{-1}]$ is invertible, then for any $\lambda \in k$, $\psi(\lambda, x_1) \neq 0$, $f_{ii}(\lambda, x_1, \bar{x}_1) \in k[x_1, \bar{x}_1, \psi(\lambda, x_1)^{-1}\psi(\lambda, \bar{x}_1)^{-1}]$ is invertible. The proof is finished.

The second possibility of δ^1 is that there is some n_2 with $n_1 < n_2 \leq n$, such that

$$\begin{cases} \delta^1(a_{n_1+1}) = K_{n_1+1} + f_{n_1+1, n_1+1}(x, x_1, \bar{x}_1)w_{n_1+1}, \\ \dots\dots\dots \\ \delta^1(a_{n_2-1}) = K_{n_2-1} + f_{n_2-1, n_1+1}(x, x_1, \bar{x}_1)w_{n_1+1} + \dots + f_{n_2-1, n_2-1}(x, x_1, \bar{x}_1)w_{n_2-1}, \\ \delta^1(a_{n_2}) = K_{n_2} + f_{n_2, n_1+1}(x, x_1, \bar{x}_1)w_{n_1+1} + \dots + f_{n_2, n_2-1}(x, x_1, \bar{x}_1)w_{n_2-1} + f_{n_2, n_2}(x, x_1, \bar{x}_1)\bar{w}' \end{cases} \quad (3.3-10)$$

with $K_i = \sum_{j=1}^{n_1-1} f_{ij}(x, x_1, \bar{x}_1)w_j$ for $i = n_1+1, \dots, n_2$, where $f_{ii}(x, x_1, x_1) \neq 0$, for $n_1 < i < n_2$; $\bar{w}' = 0$, or $\bar{w}' \neq 0$ but $f_{n_2, n_2}(x, x_1, x_1) = 0$. Define a polynomial

$$\psi_1(x, x_1) = \begin{cases} \phi(x) \prod_{i=n_1+1}^{n_2-1} c_i(x, x_1) f_{ii}(x, x_1, x_1), & \text{if } \bar{w}' = 0; \\ c_{n_2}(x, x_1) \phi(x) \prod_{i=n_1+1}^{n_2-1} c_i(x, x_1) f_{ii}(x, x_1, x_1), & \text{if } \bar{w}' \neq 0. \end{cases} \quad (3.3-11)$$

$f_{ii}(x, x_1, x_1)$ are invertible in $k[x, x_1, \psi_1(x, x_1)^{-1}]$.

Suppose we meet a boc \mathfrak{B} , whose differential is given by Formula (3.3-3) and (3.3-10), with a polynomial $\psi_1(x, x_1)$ below (3.3-11). Fix any $\lambda_0 \in k$ with $\psi_1(\lambda_0, x_1) \neq 0$, there is an induced boc $\mathfrak{B}_{(\lambda_0)}$ with $R_{(\lambda_0)} = k[x_1, \psi_1(\lambda_0, x_1)^{-1}]$ given by an unraveling $x \mapsto (\lambda_0)$, and then a series of regularization $a_i \mapsto \emptyset, w_i = 0$ for $i = 1, \dots, n_1 - 1$. We obtain three cases:

① $\mathfrak{B}_{(\lambda_0)}$ is in the case of Lemma 3.3.4 (i), then there exists some λ_1 , after sending $x_1 \mapsto (\lambda_1)$ and a series of regularization, the induced boc $\mathfrak{B}_{(\lambda_0, \lambda_1)}$ satisfies Lemma 3.3.3 (ii);

② $\mathfrak{B}_{(\lambda_0)}$ is in the case of Lemma 3.3.4 (iii);

③ $\mathfrak{B}_{(\lambda_0)}$ is in the case of Formulae (3.3-3) and (3.3-10).

In the case ③, we repeat the above procedure once again for $\mathfrak{B}_{(\lambda_0)}$. By induction on the indices of the finitely many solid arrows, we finally reach case ① or ②.

Classification 3.3.5 Let \mathfrak{B} be a local boc with R trivial, there exists four cases:

(i) \mathfrak{B} satisfies formula (3.3-5).

(ii) \mathfrak{B} has an induced boc \mathfrak{B}' satisfying Lemma 3.3.3 (i).

(iii) \mathfrak{B} has an induced local boc $\mathfrak{B}_{(\lambda_0, \lambda_1, \dots, \lambda_l)}$ for some $l < n$ satisfying Lemma 3.3.3 (ii).

(iv) \mathfrak{B} has an induced local boc $\mathfrak{B}_{(\lambda_0, \lambda_1, \dots, \lambda_{l-1})}$ for some $l < n$, satisfying Lemma 3.3.4 (ii).

Proof of 3.3.2 (i) Suppose we meet a two-point wild boc, if \mathfrak{B}_X or \mathfrak{B}_Y is in the case of Classification 3.3.5 (iii) or (iv), we may consider the wild induced local boc given by deletion. Therefore we assume that one of them satisfies Formula (3.3-5) and another satisfies Lemma 3.3.1 (i), or both are in the case of Lemma 3.3.3 (i). MW1 or MW2 follows by Proposition 3.3.1.

(ii) If we meet a local wild boc in the case of Classification 3.3.5 (iii), then there is an induced boc satisfying Lemma 3.3.3 (ii), we reach MW3.

(iii) If we meet a local wild boc in the case of 3.3.5 (iv), then there is an induced boc satisfying Lemma 3.3.4 (ii), we reach MW4 or MW5. The classification is completed.

3.4 Non-homogeneity of MW1-4

Through out the sub-section, let $\mathfrak{A}^0 = (R^0, \mathcal{K}^0, \mathcal{M}^0, H^0 = 0)$ be a matrix bi-module problem with trivial R^0 , and associated boc \mathfrak{B}^0 .

Proposition 3.4.1 [B1] If \mathfrak{B}^0 has an induced boc \mathfrak{B} in the case of MW1, then \mathfrak{B}^0 is non-homogeneous.

Proof (i) Let \mathfrak{B}_X be the induced local boc, $\vartheta_1 : R(\mathfrak{B}_X) \rightarrow R(\mathfrak{B})$, $\vartheta_2 : R(\mathfrak{B}) \rightarrow R(\mathfrak{B}^0)$ be two induced functors, $\vartheta = \vartheta_2\vartheta_1$. Set $\mathcal{L}_X = k \setminus \{\text{the roots of } \phi_X(x)\}$, for any $\lambda \in \mathcal{L}_X$ define a representation $S'_\lambda \in R(\mathfrak{B}_X)$ given by $(S'_\lambda)_X = k$, $S'_\lambda(x) = (\lambda)$. If \mathfrak{B}^0 is homogeneous, then there is a co-finite subset $\mathcal{L} \subseteq \mathcal{L}_X$ such that $\{\vartheta(S'_\lambda) \in R(\mathfrak{B}^0) \mid \lambda \in \mathcal{L}\}$ is a family of pairwise non-isomorphic homogeneous objects of $R(\mathfrak{B})$. By Corollary 3.2.4, there is an almost split conflation $(e'_\lambda) : S'_\lambda \xrightarrow{\iota'} E'_\lambda \xrightarrow{\pi'} S'_\lambda$ in $R(\mathfrak{B}'_X)$ with $E'(x) = J_2(\lambda)$ and $\vartheta(e'_\lambda)$ is an almost split conflation in $R(\mathfrak{B}^0)$. Fix any $\lambda \in \mathcal{L}^0$, and the conflation $(e_\lambda) = \vartheta_1(e'_\lambda)$ in $R(\mathfrak{B})$:

$$S_\lambda \xrightarrow{\iota} E_\lambda \xrightarrow{\pi} S_\lambda, \text{ with } \begin{cases} (S_\lambda)_X = k, & (S_\lambda)_Y = 0, & S_\lambda(x) = (\lambda), & \text{others zero;} \\ (E_\lambda)_X = k^2, & (E_\lambda)_Y = 0, & E_\lambda(x) = J_2(\lambda), & \text{others zero.} \end{cases}$$

Since $\vartheta_2(e_\lambda) = \vartheta(e'_\lambda)$, (e_λ) is almost split by Lemma 3.2.2 (ii).

(ii) Define a representation $L \in R(\mathfrak{B})$ given by $L_X = L_Y = k$, $L(x) = (\lambda)$ and $L(a_1) = (1)$. Let $g : L \rightarrow S_\lambda$ be a morphism with $g_X = (1)$, $g_Y = (0)$ and $g(v) = 0$ for all dotted arrow v 's. We assert that g is not a retraction. Otherwise, if there is a morphism $h : S_\lambda \rightarrow L$ such that $hg = id_{S_\lambda}$, then $h_X = (1)$ and $h_Y = (0)$. But h is a morphism implies that $(1)(1) = h_X L(a) = S_\lambda(a)h_Y = (0)(0)$, a contradiction.

(iii) There exists a lifting $\tilde{g} : L \rightarrow E_\lambda$ with $\tilde{g}\pi = g$. If $\tilde{g}_X = (a, b)$, then $\tilde{g}_X \pi_X = g_X$ yields $(a, b) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1)$, $a = 1$. \tilde{g} being a morphism from L to E_λ implies that $\tilde{g}_X E_\lambda(x) = L(x)\tilde{g}_X$, i.e., $(1, b) \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = (\lambda)(1, b)$, $(\lambda, 1 + b\lambda) = (\lambda, \lambda b)$, a contradiction. Thus \mathfrak{B}^0 is not homogeneous, the proof is finished.

Proposition 3.4.2 [B1] If \mathfrak{B}^0 has an induced boc \mathfrak{B} in the case of MW2, then \mathfrak{B}^0 is non-homogeneous.

Proof Since $f(x, y) \in k[x, y, \phi_X(x)^{-1}\phi_Y(y)^{-1}]$ is non-invertible, after dividing the dotted arrows v_j by some powers of $\phi_X(x)$ and $\phi_Y(y)$, we may assume that $f(x, y) = l(x, y)\alpha(x)\beta(y)$, where $l(x, y) \in k[x, y]$ and $\alpha(x)$ (resp. $\beta(y)$) is a product of some factors of $\phi_X(x)$ (resp. $\phi_Y(y)$), such that $(l(x, y), \phi_X(x)\phi_Y(y)) = 1$. By Bezout's theorem there is an infinite set

$$\mathcal{L}' = \{(\lambda, \mu) \in k \times k \mid l(\lambda, \mu) = 0, \phi_X(\lambda)\phi_Y(\mu) \neq 0\}.$$

Without loss of generality we may assume that $\mathcal{L}_X = \{\lambda \in k \mid (\lambda, \mu) \in \mathcal{L}'\}$ is an infinite set.

(i) as the same as the proof (i) of Theorem 3.4.1.

(ii) Define a representation $L \in R(\mathfrak{B})$ given by $L_X = k = L_Y$, $L(x) = (\lambda)$, $\lambda \in \mathcal{L}_X$; $L(y) = (\mu)$, $(\lambda, \mu) \in \mathcal{L}'$, and $L(a_1) = (1)$. Let $g : L \rightarrow S_\lambda$ be a morphism in $R(\mathfrak{B})$ with $g_X = (1)$, $g_Y = (0)$ and $g(v) = 0$ for all dotted arrow v 's, then g is not a retraction. Otherwise, if there is a morphism $h : S_\lambda \rightarrow L$ such that $hg = id_{S_\lambda}$, then $h_X = (1)$ and $h_Y = (0)$. But h is a morphism implies that $0 - 1 = S_\lambda(a)h_Y - h_X L(a) = h(\delta(a)) = f(\lambda, \mu)h(v) = 0$, a contradiction.

(iii) There exists a lifting $\tilde{g} : L \rightarrow E_\lambda$ with $\tilde{g}\pi = g$. A contradiction appears as the same as 3.4.1 (iii), which finishes the proof.

Proposition 3.4.3 [B1] If \mathfrak{B}^0 has an induced boc \mathfrak{B} in the case of MW3, then \mathfrak{B}^0 is non-homogeneous.

Proof Let $f_{11}(x, \bar{x}) = l(x, \bar{x})\alpha(x)\beta(\bar{x})$, where $l(x, \bar{x})$, $\alpha(x)$, $\beta(\bar{x})$ and \mathcal{L}' are given in the beginning of the proof of Proposition 3.4.2. Suppose $\mathcal{L}' = \{\lambda \in k \mid (\lambda, \mu) \in \mathcal{L}' \subseteq k\}$ is an infinite set, and $\vartheta : R(\mathfrak{B}) \rightarrow R(\mathfrak{B}^0)$ the induced functor.

Taken $S_\lambda \in R(\mathfrak{B})$ given by $(S_\lambda)_X = k$, $S_\lambda(x) = (\lambda)$, $\forall \lambda \in \mathcal{L}'$, then $S_\lambda(a_i) = (0)$ for $1 \leq i \leq n$, since $f_{ii}(\lambda, \lambda) \neq 0$. If \mathfrak{B}^0 is homogeneous, then there is a co-finite subset $\mathcal{L} \subseteq \mathcal{L}'$ such that $\{\vartheta(S_\lambda) \in R(\mathfrak{B}^0) \mid \lambda \in \mathcal{L}\}$ is a family of pairwise non-isomorphic homogeneous objects of \mathfrak{B}^0 . By Corollary 3.2.4, there is an almost split conflation $(e_\lambda) : S_\lambda \xrightarrow{\iota} E_\lambda \xrightarrow{\pi} S_\lambda$ in $R(\mathfrak{B})$ with $\vartheta(e_\lambda)$ is an almost split conflation in $R(\mathfrak{B}^0)$, where $(E_\lambda)_X = k^2$, $E_\lambda(x) = J_2(\lambda)$, $E_\lambda(a_i) = 0$ for $1 \leq i \leq n$ by Lemma 3.2.6 (ii).

Fix any $\lambda \in \mathcal{L}$ with $(\lambda, \mu) \in \mathcal{L}'$, then $f_{11}(\lambda, \mu) = 0$, $\phi(\lambda)\phi(\mu) \neq 0$. Define $L \in R(\mathfrak{B})$ with $L(X) = k^2$; $L(x) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $L(a_1) = J_2(0)$; and $L(a_i) = 0$ for $2 \leq i \leq n$. Let $g : L \rightarrow S_\lambda$ be a morphism with $g_X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $g(v_j) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all j , then g is not a retraction. Otherwise, there is a morphism $h : S_\lambda \rightarrow L$ with $hg = id_{S_\lambda}$, thus $h_X = (1, b)$. Set $h(v_1) = (c, d)$,

$$\begin{aligned} S_\lambda(a_1)h_X - h_X L(a_1) &= f_{11}(S_\lambda(x), L(x))h(v_1) \implies \\ 0(1, b) - (1, b)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= f_{11}(\lambda, \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix})(c, d) = (f_{11}(\lambda, \lambda)c, f_{11}(\lambda, \mu)d), \end{aligned}$$

which leads $-(0, 1) = (*, 0)$, a contradiction.

$$\begin{array}{ccc} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} & & (\lambda) \\ \downarrow & \xrightarrow{g} & \downarrow \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \curvearrowright L & \xrightleftharpoons[h]{} & S_\lambda \curvearrowright (0) \end{array}$$

Therefore there exists a lifting $\tilde{g} : L \rightarrow E_\lambda$ such that $\tilde{g}\pi = g$. Set $\tilde{g}_X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tilde{g}_X\pi_X = g_X$, yields $\tilde{g}_X = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$. On the other hand, $\tilde{g} : L \mapsto E_\lambda$ is a morphism, $\tilde{g}_X E_\lambda(x) = L(x)\tilde{g}_X$, i.e., $\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$, which leads $\begin{pmatrix} \lambda & 1+\lambda b \\ 0 & \lambda d \end{pmatrix} = \begin{pmatrix} \lambda & \lambda b \\ 0 & \mu d \end{pmatrix}$, a contradiction. Therefore \mathfrak{B}^0 is not homogeneous. The proof is finished.

Proposition 3.4.4 If \mathfrak{B}^0 has an induced boc \mathfrak{B} in the case of MW4, then \mathfrak{B}^0 is non-homogeneous.

Proof Fix some $\lambda \in k$ with $\psi(\lambda, x_1) \neq 0$, then $\mathcal{L}' = \{\mu \mid \psi(\lambda, \mu) \neq 0\} \subseteq k$ is a co-finite subset. Let $x \mapsto (\lambda)$, since $f_{ii}(\lambda, \lambda) \mid \phi(\lambda) \mid \psi(\lambda, x_1)$, $f_{ii}(\lambda, \lambda) \neq 0$ for $1 \leq i \leq n_1 - 1$. After a series of regularizations $a_i \mapsto \emptyset, w_i = 0$, we obtain an induced boc of \mathfrak{B} . Furthermore, since $f_{ii}(\lambda, x_1, \bar{x}_1) \in k[x_1, \psi(\lambda, x_1)^{-1}\psi(\lambda, \bar{x}_1)^{-1}]$ are invertible for $n_1 + 1 \leq i \leq n$, after regularizations $a_i \mapsto \emptyset, w_i = 0$ for $n_1 < i \leq n$, we obtain an induced minimal local boc \mathfrak{B}_λ and an induced functor ϑ_1 :

$$\mathfrak{B}_\lambda : x_1 \bigcirc X, \quad R_\lambda = k[x_1, \phi(\lambda, x_1)^{-1}], \quad \vartheta_1 : R(\mathfrak{B}_\lambda) \rightarrow R(\mathfrak{B}).$$

Set $\vartheta_2 : R(\mathfrak{B}) \rightarrow R(\mathfrak{B}^0)$ and $\vartheta = \vartheta_2\vartheta_1$. Let $S'_\mu \in R(\mathfrak{B}_\lambda)$ be given by $(S'_\mu)_X = k$ and $S'_\mu(x_1) = (\mu)$ for any $\mu \in \mathcal{L}'$.

If \mathfrak{B}^0 is homogeneous, there exists a co-finite subset $\mathcal{L} \subseteq \mathcal{L}'$, such that $\{\vartheta(S'_\mu) \in R(\mathfrak{B}^0) \mid \mu \in \mathcal{L}\}$ is a family of pairwise non-isomorphic homogeneous objects of \mathfrak{B}^0 . By Corollary 3.2.4, there is an almost split conflation (e'_μ) in $R(\mathfrak{B}_\lambda)$, such that $\vartheta(e'_\mu)$ is so in $R(\mathfrak{B}^0)$. Fix any $\mu \in \mathcal{L}$, consider the conflation $(e_\mu) = \vartheta_1(e'_\mu)$ in $R(\mathfrak{B})$, which is almost split by Lemma 3.2.2 (ii), denote n_1 by s for simple:

$$(e_\mu) : S_\mu \xrightarrow{\iota} E_\mu \xrightarrow{\pi} S_\mu, \quad (E_\mu)_X = k^2, E_\mu(x) = \lambda I_2, E_\mu(a_s) = J_2(\mu).$$

Define a representation L of \mathfrak{B} given by $L_X = k^2$, $L(x) = J_2(\lambda)$, $L(a_s) = \mu I_2$ and others zero. L is well defined, in fact if we make an unraveling for $x \mapsto J_2(\lambda)$, then by Lemma 3.2.6

(ii), after a sequence of regularizations $\begin{pmatrix} a_{i11} & a_{i12} \\ a_{i21} & a_{i22} \end{pmatrix}$ (splitting from a_i) $\mapsto \emptyset$; and $\begin{pmatrix} w_{i11} & w_{i12} \\ w_{i21} & w_{i22} \end{pmatrix}$ (splitting from w_i) $= 0$ for $i = 1, \dots, s-1$, we obtain an induced boc $\delta^0\left(\begin{pmatrix} a_{s11} & a_{s12} \\ a_{s21} & a_{s22} \end{pmatrix}\right) = 0$.

Let $g : L \rightarrow S_\mu$ be a morphism with $g_X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $g(v_j) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all possible j . It is obvious that g is not a retraction.

$$\begin{array}{ccc} & \begin{array}{c} J_2(\lambda) \\ \curvearrowright \end{array} & \\ \mu I_2 \curvearrowright L & \xrightarrow{\tilde{g}} & E_\lambda \begin{array}{c} \curvearrowright \lambda I_2 \\ J_2(\mu) \end{array} \end{array}$$

Thus there exists a lifting $\tilde{g} : L \rightarrow E_\lambda$ with $\tilde{g}\pi = g$. $\tilde{g}_X\pi_X = g_X$ leads to $\tilde{g}_X = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$. On the other hand, since $(x-y)^2 \mid f_{11}(x, y)$ if $\bar{w} \neq 0$, $(J_2(\lambda) - \lambda I_2)^2 = 0$ and hence $f_{11}(J_2(\lambda), \lambda I_2) = 0$. $\tilde{g} : L \rightarrow E_\lambda$ being a morphism implies that

$$\begin{aligned} L(a_1)\tilde{g}_X - \tilde{g}_X E_\lambda(a_1) &= \begin{cases} f_{11}(L(x), E_\lambda(x))\tilde{g}(\bar{w}) = 0, & \bar{w} \neq 0; \\ 0, & \bar{w} = 0, \end{cases} \\ \implies \mu \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} - \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} J_2(\mu) = 0, & \text{i.e. } -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

The contradiction tells that \mathfrak{B}^0 is non-homogeneous. The proof is completed.

Proposition 3.4.5 Let \mathfrak{B}^0 be a boc \mathfrak{B} be an induced boc with $\mathcal{T} = \{X, Y\}$; $R_X = k[x, \phi(x)^{-1}]$, $R_Y = k1_Y$; and a layer $L = (R; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$, $\delta(a_1) = 0$:

$$\mathfrak{B} : \quad x \curvearrowright X \xrightarrow{a_1} Y$$

Making a reduction given by proposition 2.2.7, we obtain an induced local boc. Suppose all the loops a_2, \dots, a_n in the induced boc are going to \emptyset by a sequence of regularizations, and some localizations before each base changes, the induced boc \mathfrak{B}' is minimal with $R' = k[x, \phi'(x)^{-1}]$. Then \mathfrak{B}^0 is non-homogeneous.

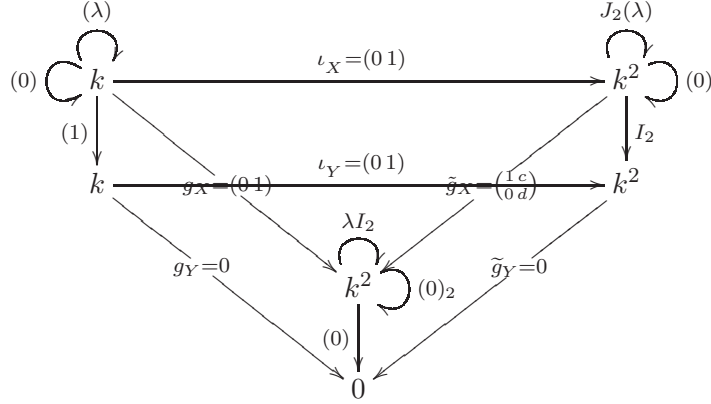
Proof (i) The infinite subset $\mathcal{L}' = \{\lambda \mid \phi'(\lambda) \neq 0\} \subseteq k$ gives a set of pairwise non-isomorphic objects $S'_\lambda \in R(\mathfrak{B}')$ with $S'_\lambda(X) = k$, $S'_\lambda(x) = (\lambda)$, $\forall \lambda \in \mathcal{L}'$. Write the induced functor $\vartheta_1 : R(\mathfrak{B}') \rightarrow R(\mathfrak{B})$, then

$$S_\lambda = \vartheta_1(S'_\lambda) : \quad S_\lambda(x)=(\lambda) \curvearrowright k \xrightarrow{S_\lambda(a_1)=(1)} k \in R(\mathfrak{B}),$$

Let $\vartheta_2 : R(\mathfrak{B}) \rightarrow R(\mathfrak{B}^0)$ be the induced functor, $\vartheta = \vartheta_2\vartheta_1$. If \mathfrak{B}^0 is homogeneous, then there is a co-finite subset $\mathcal{L} \subseteq \mathcal{L}'$ such that $\{\vartheta(S'_\lambda) \in R(\mathfrak{B}^0) \mid \lambda \in \mathcal{L}\}$ is a family of homogeneous objects. Using Corollary 3.2.5, there is an almost split conflation $(e'_\lambda) : S'_\lambda \xrightarrow{\iota} E'_\lambda \xrightarrow{\pi} S'_\lambda$ in $R(\mathfrak{B}')$ with $E'_\lambda(x) = J_2(\lambda)$, such that $\vartheta(e'_\lambda)$ is an almost split conflation in $R(\mathfrak{B}^0)$.

(ii) Fix any λ , there is an almost split conflation: $(e_\lambda) = \vartheta_1(e'_\lambda) : S_\lambda \rightarrow E_\lambda \rightarrow S_\lambda$ by lemma 3.2.2 (ii). Define an object $L' \in R(\mathfrak{B}_X)$ with $L'_X = k^2$, $L'(x) = \lambda I_2$, $L'(a_i) = (0)_{2 \times 2}$ for all $a_i : X \mapsto X$. Set the induced functor $\vartheta_3 : R(\mathfrak{B}_X) \rightarrow R(\mathfrak{B})$ and $L = \vartheta_3(L')$, then $L(a_i) = 0$, $1 \leq i \leq n$. Define a morphism $g : S_\lambda \rightarrow L$ with $g_X = (0 \ 1)$, $g_Y = 0$, and $g(v) = 0$ for any dotted arrows. We claim that g is not a retraction. Otherwise, if there is a morphism $h : L \rightarrow S_\lambda$ with $gh = id_{S_\lambda}$, then $h_X = \begin{pmatrix} 1 \\ c \end{pmatrix}$. Since $h_X S(a_1) = L(a_1)h_Y$, i.e. $\begin{pmatrix} 1 \\ c \end{pmatrix}(1) = 0$, a

contradiction



Therefore, there exists a lifting $\tilde{g} : L \rightarrow E_\lambda$ with $\iota\tilde{g} = g$, $\tilde{g}_X = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Since \tilde{g} is a morphism, we have $E(x)\tilde{g}_X = \tilde{g}_X L(x)$, i.e. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, a contradiction. Therefore \mathfrak{B}^0 is not homogeneous. The proof is completed.

4. One-sided pairs

Through out the present section, we always assume that $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ is a bipartite matrix bi-module problem with RDCC condition given by Remark 1.4.4. We will define some special quotient problem of some induced bi-module problem of \mathfrak{A} .

4.1 Definition of one-sided pairs

Let $\mathfrak{A}^0 = (R^0, \mathcal{K}^0, \mathcal{M}^0, H^0 = 0)$ be a bipartite matrix bi-module problem with RDCC condition and the corresponding bi-co-module problem \mathfrak{C}^0 and boc \mathfrak{B}^0 . Let a sequence of pairs

$$(\mathfrak{A}^0, \mathfrak{B}^0), (\mathfrak{A}^1, \mathfrak{B}^1), \dots, (\mathfrak{A}^r, \mathfrak{B}^r)$$

be given by reductions in the sense of Lemma 2.3.2 at each step. Suppose that the leading position of the first base matrix A_1^r of \mathcal{M}_1^r is $(p^r, q^r + 1)$ over \mathcal{T}^r , which is sitting at the leading block $(\mathfrak{p}, \mathfrak{q})$ of a certain base matrix of \mathcal{M}_1^0 partitioned by \mathcal{T}^0 . We further assume that d_1, \dots, d_m are the first m solid arrows of \mathfrak{B}^r , which locate at the p -th row in the formal product Θ^r , such that d_m is sitting at the last column of the $(\mathfrak{p}, \mathfrak{q})$ -block, see the picture below:

$$\begin{array}{|c|} \hline \begin{array}{|c|c|c|} \hline d_1 & \cdots & d_m \end{array} \\ \hline \end{array} \quad (4.1-1)$$

Recall the notation below Formulae (2.1-2): we have the quotient problem of the matrix bi-module problem $(\mathfrak{A}^r)^{[m]} = (R^r, \mathcal{K}^r, (\mathcal{M}^r)^{[m]}, H^r)$ of \mathfrak{A}^r ; the sub-co-bi-module problem $(\mathfrak{C}^r)^{(m)} = (R^r, \mathcal{C}^r, (\mathcal{N}^r)^{(m)}, \partial |_{(\mathcal{N}^r)^{(m)}})$ of \mathfrak{C}^r with the quasi-basis d_1, \dots, d_m of $(\mathcal{N}^r)^{(m)}$; and the sub-boc $(\mathfrak{B}^r)^{(m)}$ of \mathfrak{B}^r . We obtain a *quotient-sub-pair* $((\mathfrak{A}^r)^{[m]}, (\mathfrak{B}^r)^{(m)})$. Denote by $\bar{A}_1^r, \dots, \bar{A}_m^r$ the quasi-basis of $(\mathcal{M}^r)^{[m]}$ over $R^r \otimes_k R^r$, we have the formal equation of the pair:

$$\begin{aligned} & (\sum_{Y \in \mathcal{T}^r} e_Y * E_Y + \sum_{j=1}^{t^r} v_j^r * V_j^r) (\sum_{i=1}^m d_i * \bar{A}_i^r + H^r) \\ &= (\sum_{i=1}^m d_i * \bar{A}_i^r + H^r) (\sum_{Y \in \mathcal{T}^r} e_Y * E_Y + \sum_{j=1}^{t^r} v_j^r * V_j^r). \end{aligned}$$

Let $d_j : X \rightarrow Y_j$ (possibly $Y_j = X$), which can be rewritten as a *reduced formal equation*:

$$e_X(d_1, d_2, \dots, d_m) = (w_1, w_2, \dots, w_m) + (d_1, d_2, \dots, d_m) \begin{pmatrix} e_{Y_1} & w_{12} & \cdots & w_{1m} \\ & e_{Y_2} & \cdots & w_{2m} \\ & & \ddots & \vdots \\ & & & e_{Y_m} \end{pmatrix} \quad (4.1-2)$$

Remark 4.1.1 (i) In the formula (4.1-2), e_X is the (p, p) -th entry of the formal product $\sum_{Y \in \mathcal{T}^r} e_Y * E_Y$; and e_{Y_ξ} the $(q + \xi, q + \xi)$ -th entry of that for $\xi = 1, \dots, m$.

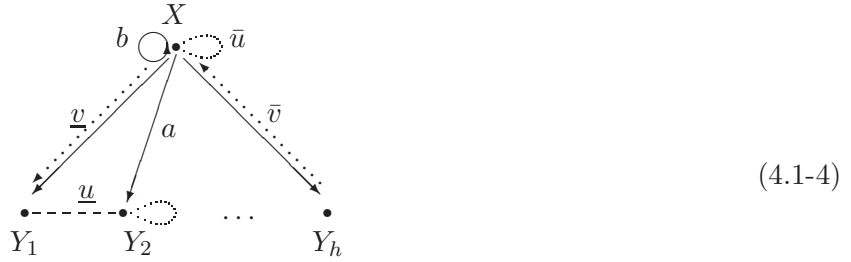
(ii) (w_1, \dots, w_m) is the $(p, q + 1)$ -th, \dots , $(p, q + m)$ -th entries of $(\sum_{j=1}^{t^r} v_j^r * V_j^r) H^r - H^r (\sum_{j=1}^{t^r} v_j^r * V_j^r)$. $w_\xi = \sum_j \alpha_\xi^j v_j^r$, where j runs over $s(v_j^r) \ni p, e(v_j^r) \ni q + \xi, \alpha_\xi^j \in k, 1 \leq \xi \leq m$.

(iii) For $1 \leq \eta < \xi \leq m$, $w_{\eta\xi}$ is the $(q + \eta, q + \xi)$ -th entry of $\sum_{j=1}^{t^r} v_j^r * V_j^r$. And $w_{\eta\xi} = \sum_j \beta_{\eta\xi}^j v_j^r$ where j runs over $s(v_j^r) \ni q + \eta, e(v_j^r) \ni q + \xi, \beta_{\eta\xi}^j \in k$.

(iv) The differential of any solid arrows can be read off from the reduced formal product, we notice that the solid arrows appear in each monomial only once from the left to a dotted arrow:

$$-\delta(d_i) = w_i + \sum_{j < i} d_j w_{ij}, \quad 1 \leq i \leq m. \quad (4.1-3)$$

Definition 4.1.2 A boc \mathfrak{B} with a layer $L = (R; \omega; a_1, \dots, a_m, b_1, \dots, b_n; \underline{u}_j, \underline{v}_j, \bar{u}_j, \bar{v}_j)$, see the picture below, is called *one-sided*, provided R is trivial; $\delta(a_i) = \sum_j \alpha_{ij} \underline{v}_j + \sum_{b_{i'} \prec a_i, j} \beta_{ii'} b_{i'} \underline{v}_j + \sum_{i' < i, j} \gamma_{ii'} a_{i'} \underline{u}_j$, $\delta(b_i) = \sum_j \lambda_{ij} \bar{u}_j + \sum_{i' < i, j} \mu_{ii'} b_{i'} \bar{u}_j + \sum_{a_{i'} \prec b_i, j} \nu_{ii'} a_{i'} \bar{v}_j$ with constant coefficients.



The associated boc $(\mathfrak{B}^r)^{(m)}$ of $(\mathfrak{A}^r)^{[m]}$ is one sided by Formula (4.1-3). We call this quotient-sub-pair a *one-sided pair*, and denoted by $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ for simple.

Write $\bar{\mathfrak{A}} = (R, \mathcal{K}, \bar{\mathcal{M}}, F)$, where $R = k1_X \times \prod_{j=1}^h k1_{Y_j}$ is a trivial sub-algebra of R^r ; $\{X; Y_1, \dots, Y_h\} \subseteq \mathcal{T}^r$; $\bar{\mathcal{M}}_1$ has an R - R -quasi-basis $(1_{XY_1(d_1)}, 0, \dots, 0), \dots, (0, 0, \dots, 1_{XY_1(d_m)})$; $F = (0)$; let $E_X = (1_X) \in \text{IM}_{1 \times 1}(R)$, and $E_{Y_j} = \sum_{p \in Y_j} 1_{Y_j} E_{pp}$ with E_{pq} the matrix units of $\text{IM}_{m \times m}(R)$, then $\{E_X; E_{Y_j}\}_j$ forms a part of the R -quasi-basis of \mathcal{K}_0 . However, there exist some linear relations between w_i and w_{ij} in the formula (4.1-2).

Now we start the reduction procedure in the sense of Lemma 2.3.2. Let $(\mathfrak{A}, \mathfrak{B})$ be any pair, $(\mathfrak{A}^{[p]}, \mathfrak{B}^{(p)})$ be a quotient-sub-pair. Since the reduction for any pair is made with respect to an admissible \bar{R} - R' -bi-module L by Proposition 2.2.1-2.2.7, or a regularization in Proposition 2.2.8, which is completely as the same as to make a reduction for the quotient-sub-pair. Therefore there are two sequences of reductions:

$$\begin{aligned} &(\bar{\mathfrak{A}}, \bar{\mathfrak{B}}), \quad (\bar{\mathfrak{A}}^1, \bar{\mathfrak{B}}^1), \quad \dots, \quad (\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i) \quad \dots, \quad (\bar{\mathfrak{A}}^s, \bar{\mathfrak{B}}^s); \\ &(\mathfrak{A}^r, \mathfrak{B}^r), \quad (\mathfrak{A}^{r+1}, \mathfrak{B}^{r+1}), \quad \dots, \quad (\mathfrak{A}^{r+i}, \mathfrak{B}^{r+i}), \quad \dots, \quad (\mathfrak{A}^{r+s}, \mathfrak{B}^{r+s}). \end{aligned} \quad (4.1-5)$$

From now on, we perform reductions according to Formula (2.4-3). More precisely, the system \mathbb{F}^{r_i} in (2.4-3) for the pair $(\mathfrak{A}^{r+i}, \mathfrak{B}^{r+i})$ can be written as a reduced form $\bar{\mathbb{F}}^i$ for the pair $(\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i)$:

$$\bar{\mathbb{F}}^i : \bar{\Psi}_{\underline{n}}^l F^i \equiv \prec(\bar{p}^i, \bar{q}^i) \bar{\Psi}_{\underline{n}}^m + F^i \bar{\Psi}_{\underline{n}}^r, \quad (4.1-6)$$

where the upper indices l, m, r on $\bar{\Psi}$ indicate left, middle and right parts of the matrices of dotted elements. Since it is difficult to determine the dotted arrows after a reduction, instead we describe the linear relation of the dotted elements in $\bar{\Psi}$ appearing at the reduction. In order to do so, we define a *pseudo reduced formal equation* of the pair $(\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i)$:

$$e_x^i(F^i + \bar{\Theta}^i) = (W_1, \dots, W_m) + (F^i + \bar{\Theta}^i) \begin{pmatrix} e_{Y_1}^i & W_{12} & \cdots & W_{1m} \\ & e_{Y_2}^i & \cdots & W_{2m} \\ & & \ddots & \vdots \\ & & & e_{Y_m}^i \end{pmatrix}, \quad (4.1-7)$$

where W_h and W_{hl} split from w_h, w_{hl} ; e_Z^i is given by the proof (ii) of Theorem 2.4.1. $\bar{\Theta}^i$ is the reduced formal product of $(\bar{\mathcal{M}}_1^i, \bar{\mathcal{N}}_1^i)$. $F^i + \bar{\Theta}^i$ is an $(1 \times \underline{m})$ -partitioned matrix under $\bar{\mathcal{T}}$ with size vector $\underline{n}^i = (n_0^i; n_1^i, \dots, n_m^i)$ as shown below, where F^i is sitting in the blank part:

$$F^i + \bar{\Theta}^i = \begin{array}{|c|c|c|c|c|c|} \hline & & d_q^i & \cdots & \cdots & \cdots \\ \hline & & & & d_p^i & d_{p+1}^i \cdots \\ \hline & & & & d_1^i & d_2^i \cdots \\ \hline & & & & & \cdots d_{p-1}^i \\ \hline & & & & & \\ \hline \end{array}$$

Remark 4.1.3 Let $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$ be any induced pair of $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ after several reductions in the sense of Lemma 2.3.2. And $(\bar{\mathfrak{A}}'', \bar{\mathfrak{B}}'')$ is an induced pair of $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$ by one reduction of 2.3.2.

(i) If we have a linear relation of dotted elements $\sum_j u_j = 0$ in $\bar{\mathbb{F}}'$, then $\sum_j \bar{u}_j = 0$ in $\bar{\mathbb{F}}''$ with \bar{u}_j being the split of u_j .

(ii) Suppose the first arrow $\delta(a'_1) = v + \sum_j \alpha_j u_j$ in $\bar{\mathfrak{B}}'$, where v, u_j are dotted elements of $\bar{\Psi}_{\underline{n}'}$. If v is a dotted arrow, and $v \notin \{u_j\}$, then $\delta(a'_1) \neq 0$.

(iii) Set $a'_1 \mapsto \emptyset$ given above, we sometimes say that v is *replaced by* $-\sum_j u_j$ in $\bar{\Psi}_{\underline{n}''}$.

(iv) If we are able to determine, that a dotted element v of $\bar{\Psi}_{\underline{n}''}$ is linearly independent of all the others, then v is said to be a *dotted arrow preserved in* $\bar{\mathfrak{B}}''$.

We are able to read off the differential δ of the solid arrows from the pseudo reduced formal equation by Theorem 1.6.4. Denote the (p, q) -th entry of F^i splitting from $d_l : X \mapsto Y$ by f_{lpq}^i :

$$\delta(d_{l\bar{p}\bar{q}}^i) = -w_{lpq}^i - \sum_{j>l, q<\bar{q}} f_{l\bar{p}q}^i w_{lj, q\bar{q}}^i - \sum_{q<\bar{q}} f_{l\bar{p}q}^i w_{Yq\bar{q}}^i + \sum_{q, j<l} w_{X\bar{p}q}^i f_{lq\bar{q}}^i. \quad (4.1-8)$$

4.2 The differentials in one sided pairs

Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair given in formulae (4.1-2) and (4.1-4). This subsection is devoted to calculating the differentials of the solid arrows of $\bar{\mathfrak{B}}$.

Suppose first that $\bar{\mathfrak{B}}$ is a one sided local bocs having a layer $L = (R; \omega; b_1, \dots, b_n; v_1, \dots, v_m)$ with $R = k1_X$. Recall Classification 3.3.2, which is simpler in one sided case.

Classification 4.2.1 Let $\bar{\mathfrak{B}}$ be a one sided local bocs given above.

(i) $\bar{\mathfrak{B}}$ satisfies the triangular formula (3.3-5) (letter a is changed to letter b). After making an easy base change, (3.3-5) can be written as: $\delta^0(b_1) = \bar{u}_1, \dots, \delta^0(b_n) = \bar{u}_n$.

Formulae (3.3-6) can be written as following given by a base change,

$$\begin{cases} \delta^0(b_1) = \bar{u}_1, \dots \delta^0(b_{n_0-1}) = \bar{u}_{n_0-1}, \\ \delta^0(b_{n_0}) = \alpha_{n_0,1} \bar{u}_1 + \dots + \alpha_{n_0, n_0-1} \bar{u}_{n_0-1}, \end{cases} \quad (4.2-1)$$

with $\alpha_{n_0,j} \in k$. After a series of regularization, Formula (3.3-2) with $x = b_{n_0}$ can be written as:

$$\begin{cases} \delta^0(b_{n_0+1}) &= h_{n_0+1,1}(x)\bar{u}_{n_0+1}, \\ \delta^0(b_{n_0+2}) &= h_{n_0+2,1}(x)\bar{u}_{n_0+1} + h_{n_0+2,2}(x)\bar{u}_{n_0+2}, \\ &\dots \dots \\ \delta^0(b_n) &= h_{n,n_0+1}(x)\bar{u}_{n_0+1} + h_{n,n_0+2}(x)\bar{u}_{n_0+2} + \dots + h_{n,n}(x)\bar{u}_n, \end{cases} \quad (4.2-2)$$

with the polynomial $\phi = 1$, and $h_{n_0+i,j}(x) = \alpha_{n_0+i,j}^0 + \alpha_{n_0+i,j}^1 x$ of degree 1.

(ii) $\bar{\mathfrak{B}}$ satisfies Lemma 3.3.3 (i), if $h_{ii}(x) \in k \setminus \{0\}$ for $n_0 < i \leq n$.

(iii) $\bar{\mathfrak{B}}$ has an induced boc $\bar{\mathfrak{B}}_{(\lambda_0, \dots, \lambda_l)}$ satisfies Lemma 3.3.3 (ii), if $h_{ss}(x) \in k[x] \setminus k$ non-invertible for some $n_0 < s \leq n$, which is in the case of MW3.

(iv) $\bar{\mathfrak{B}}$ has an induced boc $\bar{\mathfrak{B}}_{(\lambda_0, \dots, \lambda_{l-1})}$ for some $l < n$, which satisfies Lemma 3.3.4 (ii) with a triangular formula similar to Formula (4.2-2) for b_1, \dots, b_{n_1} with a polynomial $\phi(x) = \prod_{i=1}^{n_1-1} h_{ii}(x) \neq 0$, and $h_{n_1,n_1}(x) = 0$, i.e. $\bar{w} = 0$. Moreover the differential δ^1 with respect to $x, x_1 = b_{n_1}$, and a polynomial $\psi(x) = \phi(x) \prod_{i=n_1+1}^n c_i(x) \neq 0$ given by Formula (3.3-4):

$$\begin{cases} \delta^1(b_{n_1+1}) &= K_{n_1+1} + h_{n_1+1,n_1+1}(x)u_{n_1+1}, \\ &\dots \\ \delta^1(b_n) &= K_n + h_{n,n_1+1}(x, x_1)u_{n_1+1} + \dots + h_{nn}(x)u_n, \end{cases} \quad (4.2-3)$$

where $h_{ii}(x) \mid \psi(x)$ for $i = n_1 + 1, \dots, n$. Then $\bar{\mathfrak{B}}_{(\lambda_0, \dots, \lambda_{l-1})}$ is in the case of MW4 with $\bar{w} = 0$. In particular the case of MW5 can not occur.

Now we concern the general one sided pairs $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$. Suppose $\bar{\mathfrak{B}}_X$, the induced local boc of $\bar{\mathfrak{B}}$, is in the case of Classification 4.2.1 (iii) or (iv), then $\bar{\mathfrak{B}}_X$ is wild and non-homogeneous, so is $\bar{\mathfrak{B}}$. Since $\bar{\mathfrak{B}}_X$ satisfies 4.2.1 (i) is relatively simple, we now concentrate on the case of 4.2.1 (ii). Denote the solid edges of $\bar{\mathfrak{B}}$ before b_{n_0} by a_1, \dots, a_h , the differential δ^0 on a 's can be written as:

$$\delta^0(a_1) = \underline{v}_1, \dots, \delta^0(a_h) = \underline{v}_h; \quad (4.2-4)$$

or $\delta^0(a_1) = \underline{v}_1, \dots, \delta^0(a_{h_1-1}) = \underline{v}_{h_1-1}, \delta^0(a_{h_1}) = \sum_{j=1}^{h_1-1} \beta_{h_1j} \underline{v}_j$; then $\delta^0(a_{h_1+1}) = \underline{v}_{h_1+1}, \dots, \delta^0(a_{h_2-1}) = \underline{v}_{h_2-1}, \delta^0(a_{h_2}) = \sum_{j \neq n_1, j=1}^{h_2-1} \beta_{h_2j} \underline{v}_j$; continuously we obtain say s formulae with $\beta \in k$ and s edges $a_{h_1}, a_{h_2}, \dots, a_{h_s}$. Set $\Lambda = \{1, \dots, h\} \setminus \{h_1, \dots, h_s\}$, then

$$\begin{cases} \delta^0(a_i) &= \underline{v}_i, & i \in \Lambda; \\ \delta^0(a_{h_l}) &= \sum_{j \in \Lambda, j < h_l} \beta_{h_lj} \underline{v}_j, & l = 1, \dots, s. \end{cases} \quad (4.2-5)$$

Convention 4.2.2 Suppose $\bar{\mathfrak{B}}$ is a one-sided boc with $\bar{\mathfrak{B}}_X$ satisfying 4.2.1 (ii). ① All the loops b_1, \dots, b_n at X are called *b-class arrows*, where the loop $\bar{b} = b_{n_0}$ is said to be *effective* or *\bar{b} -class*; the others are *non-effective*. ② The edges a_1, \dots, a_h before \bar{b} are called *a-class arrows*, where $\{\bar{a}_i = a_{h_i} \mid 1 \leq i \leq s\}$ are said to be *effective* or *\bar{a} -class*; the others are *non-effective*. ③ Let c_1, c_2, \dots, c_t be the solid edges after \bar{b} , which are called *c-class arrows* and *effective*.

An solid arrow splitting from one of the classes $a, \bar{a}; c; b, \bar{b}$, and a dotted element from $\underline{u}, \underline{v}, \bar{u}, \bar{v}$ is said to be the same class.

Next we give a special case of the differentials δ^1 with respect to \bar{b} on c -class arrows, where $\{\underline{v}_j\}_{j \in \Lambda} \cup \{\underline{v}_{h+j}\}_{1 \leq j \leq t}$ are dotted arrows; $\gamma_{ij}(\bar{b}) = \gamma_{ij}^0 + \gamma_{ij}^1 \bar{b}, \gamma_{ij}^0, \gamma_{ij}^1 \in k; \gamma_{i,h+i}(\bar{b}) \neq 0, 1 \leq i \leq t$:

$$\begin{cases} \delta^1(c_1) = \sum_{j \in \Lambda} \gamma_{1j}(\bar{b}) \underline{v}_j + \gamma_{1,h+1}(\bar{b}) \underline{v}_{h+1}, \\ \dots \dots \dots \\ \delta^1(c_t) = \sum_{j \in \Lambda} \gamma_{tj}(\bar{b}) \underline{v}_j + \gamma_{t,h+1}(\bar{b}) \underline{v}_{h+1} + \dots + \gamma_{t,h+t}(\bar{b}) \underline{v}_{h+t}, \end{cases} \quad (4.2-6)$$

Lemma 4.2.3 Let $\bar{\mathfrak{B}}$ be a one-sided boc with $\bar{\mathfrak{B}}_X$ satisfying 4.2.1 (ii). If Formula (4.2-6) fails, i.e. there exists some $1 \leq t_1 \leq t$ with $\gamma_{t_1, h+t_1} = 0$, then $\bar{\mathfrak{B}}$ is not homogeneous.

Proof We make a sequence of regularization $a_j \mapsto \emptyset, \underline{v}_j = 0$ for $j \in \Lambda$, $b_j \mapsto \emptyset, \bar{u}_j = 0$ for $1 \leq j < n_0$, and edge reduction $\bar{a}_i \mapsto (0)$ for $i = 1, \dots, s$, then obtain an induced pair $(\mathfrak{A}', \mathfrak{B}')$. Without loss of generality, suppose $\mathcal{T}' = \{X, Y\}$. Make a loop mutation $\bar{b} \mapsto (x)$, a localization $\phi(x) = \prod_{i=1}^{t_1-1} g_{i, h+i}(x)$, and regularizations $c_i \mapsto \emptyset, \underline{v}'_{h+i} = 0$ for $1 \leq i < t_1$, we obtain an induced boc $\mathfrak{C} \xrightarrow{c_{t_1}} \cdot$ of two vertices with $\delta(c_{t_1}) = 0$, which is in the case of MW1. Thus \mathfrak{B}' , consequently $\bar{\mathfrak{B}}$, is wild and non-homogeneous, this finishes the proof.

Let $\bar{\delta}$ be obtained from the differential δ by removing all the monomial involving some non-effective a, b -class solid arrows. Now we write $\bar{\delta}$ acting on all a, b, c -class arrows in the following three formulae (where the third one is given under the assumption that the differential δ^1 of c -class arrows satisfies Formula (4.2-6)):

$$\begin{aligned} \bar{\delta}(a_i) &= \underline{v}_i + \sum_{h_l < i} \bar{a}_l (\sum_j \epsilon_{ilj} \underline{u}_j), & \epsilon_{ilj} \in k, i \in \Lambda; \\ \bar{\delta}(\bar{a}_\tau) &= \sum_{j < h_\tau} \bar{\beta}_{\tau j} \underline{v}_j + \sum_{l < \tau} \bar{a}_l (\sum_j \bar{\epsilon}_{\tau l j} \underline{u}_j), & \bar{\beta}_{\tau j}, \bar{\epsilon}_{\tau l j} \in k, 1 \leq \tau \leq s. \end{aligned} \quad (4.2-7)$$

$$\begin{aligned} \bar{\delta}(b_i) &= \bar{u}_i + \sum_{\bar{a}_l < b_i} \bar{a}_l (\sum_j \epsilon_{ilj} \bar{v}_j), & \epsilon_{ilj} \in k, i < n_0; \\ \bar{\delta}(\bar{b}) &= \sum_{j=1}^{n_0-1} \bar{\alpha}_j \bar{u}_j + \sum_{l=1}^s \bar{a}_l (\sum_j \bar{\epsilon}_{l j} \bar{v}_j), & \bar{\alpha}_j, \bar{\epsilon}_{l j} \in k, i = n_0; \\ \bar{\delta}(b_i) &= \bar{u}_i + \sum_{j=1, j \neq n_0}^{i-1} \alpha_{ij}(\bar{b}) \bar{u}_j + \sum_{l=1}^s \bar{a}_l (\sum_j \epsilon_{ilj} \bar{v}_j) + \sum_{c_l < b_i} c_l (\sum_j \epsilon'_{ilj} \bar{v}_j), \\ & \alpha_{ij}(\bar{b}) \in k[\bar{b}], \quad \epsilon_{ilj}, \epsilon'_{ilj} \in k, \quad i > n_0. \end{aligned} \quad (4.2-8)$$

$$\bar{\delta}(c_\tau) = \sum_{l=1}^s \bar{a}_l (\sum_j \zeta_{\tau l j} \underline{u}_j) + \sum_{j \leq h_\tau} \gamma_{\tau j}(\bar{b}) \underline{v}_j + \sum_{l=1}^{i-1} c_l (\sum_j \xi_{\tau l j} \underline{u}_j), \quad \zeta_{\tau l j}, \xi_{\tau l j} \in k. \quad (4.2-9)$$

4.3. Reduction sequences of one-sided pairs

In the present subsection, we will construct a reduction sequence starting from a one-sided pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ with at least two vertices, where $\bar{\mathfrak{B}}_X$ satisfies Classification 4.2.1 (ii).

Condition 4.3.1 (BRC) Let $(\mathfrak{A}, \mathfrak{B})$ with trivial \mathcal{T} be any pair of a matrix bi-module problem and the associated boc.

(i) Suppose the solid arrows $\mathcal{D} = \{d_1, \dots, d_q\}$ and $\mathcal{E} = \{e_1, \dots, e_p\}$ locate at the bottom row of Θ form the first $p+q$ arrows of \mathfrak{B} (not necessarily full of the row), such that e_1, \dots, e_{p-1} are edges starting from X , $e = e_p$ is a loop at X , and $d_i < e_p, 1 \leq i \leq q$. There exists a set of dotted arrows $\mathcal{U} = \{u_1, \dots, u_q\}$, write $\mathcal{W} = \mathcal{V}^* \setminus \mathcal{U} = \{w_1, \dots, w_t\}$.

(ii) Denote by $\bar{\delta}$ the part of the differential of any solid arrow by removing all the monomials containing any solid arrow besides of $\{e_1, \dots, e_p\}$, such that

$$\begin{aligned} \bar{\delta}(d_i) &= u_i + \sum_{j=1}^t (\sum_{e_l < d_i} \lambda_{ijl} e_l) w_j, & \lambda_{ijl} \in k, i = 1, \dots, q; \\ \bar{\delta}(e_i) &= \sum_{d_j < e_i} \alpha_{ij} u_j + \sum_{j=1}^t (\sum_{l=1}^{i-1} \mu_{ijl} e_l) w_j, & \alpha_{ij}, \mu_{ijl} \in k, i = 1, \dots, p. \end{aligned}$$

Then $(\mathfrak{A}, \mathfrak{B})$ is said to have the *bottom row condition* (BRC) with respect to $(\mathcal{D}, \mathcal{U})$ and $(\mathcal{E}, \mathcal{W})$.

Suppose the pair $(\mathfrak{A}, \mathfrak{B})$ satisfies (BRC) with $p > 1$ or $q > 1$ and the first arrow $a_1 : X \mapsto Y$, where $X \neq Y$ if $a_1 = e_1$. Now we discuss the condition (BRC) on the induced pair $(\mathfrak{A}', \mathfrak{B}')$.

- (i) If $a_1 = d_1$, then $d_1 \mapsto \emptyset$, let $\mathcal{D}' = \{d_2, \dots, d_q\}, \mathcal{U}' = \{u_2, \dots, u_q\}$ and $\mathcal{E}' = \mathcal{E}, \mathcal{W}' = \mathcal{W}$.
- (ii) If $a_1 = e_1$, and $e_1 \mapsto (0)$, let $\mathcal{D}' = \mathcal{D}, \mathcal{U}' = \mathcal{U}; \mathcal{E}' = \{e_2, \dots, e_p\}, \mathcal{W}' = \mathcal{W}$.
- (iii) If $a_1 = e_1, e_1 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, let $\mathcal{D}' = \{d_{i21}, d_{i22} \mid t(d_i) = X\} \cup \{d_{j2} \mid t(d_j) \neq X\}$, correspondingly $\mathcal{U}' = \{u_{i21}, u_{i22}, u_{j2}\}; \mathcal{E}' = \{e_{22}, \dots, e_{p-1,2}, e_{p21}, e_{p22}\}$, where \bullet_{i2} , resp. \bullet_{i2l} , stands for the arrow of \mathfrak{B}' locating at the second row of the 2×1 , resp. 2×2 , matrix splitting from \bullet_i of \mathfrak{B} .
- (iv) If $a_1 = e_1, e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, set $\mathcal{D}' = \{d_{i21}, d_{i22} \mid t(d_i) = X \text{ or } Y\} \cup \{d_{j2} \mid t(d_j) \neq X, Y\}$, $\mathcal{U} = \{u_{i21}, u_{i22}, u_{j2}\}; \mathcal{E}' = \{e_{i21}, e_{i22} \mid t(e_i) = Y\} \cup \{e_{j2} \mid t(e_j) \neq Y\} \cup \{e_{p21}, e_{p22}\}$.

Lemma 4.3.2 Suppose the pair $(\mathfrak{A}, \mathfrak{B})$ satisfies (BRC) with $p > 1$ or $q > 1$ and the first arrow $a_1 : X \mapsto Y$. Then after making a reduction $a_1 \mapsto G$ as above (i)-(iv), the induced pair $(\mathfrak{A}', \mathfrak{B}')$ satisfies (BRC) with respect to $(\mathcal{D}', \mathcal{U}')$ and $(\mathcal{E}', \mathcal{W}')$ defined as above (i)-(iv).

Proof The case (i) and (ii) are trivial. Suppose $G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in (iii) resp. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in (iv), then X, Y split into two vertices X', Y' resp. three vertices X', Y', Y'' :

$$e_X \mapsto e'_X = \begin{pmatrix} e_{Y'} & w \\ & e_{X'} \end{pmatrix}, \quad e_Y \mapsto e'_Y = e_{Y'} \quad \text{resp.} \quad e_Y \mapsto e'_Y = \begin{pmatrix} e_{Y''} & w' \\ & e_{Y'} \end{pmatrix}.$$

In the two cases, e_{i2} or e_{i21}, e_{i22} for $1 < i < p$ starting at X' do not end at X' , since $t(e_i) \neq X'$ by (RBC) (i) on $(\mathfrak{A}, \mathfrak{B})$; the edge $e_{p21} : X' \mapsto Y'$, and the loop $e_{p22} : X' \mapsto X'$. By (BRC)(ii),

$$\begin{aligned} \bar{\delta}(D_i) &= U_i + \sum_{j=1}^t \lambda_{ij1} G W_j + \sum_{j=1}^t (\sum_{e_l \prec d_i} \lambda_{ijl} E_l) w_j, \\ \bar{\delta}(E_i) &= \sum_{d_j \prec e_i} \alpha_{ij} U_j + \sum_{j=1}^t \mu_{ij1} G W_j + \sum_{j=1}^t (\sum_{l=2}^{i-1} \mu_{ijl} E_l) W_j + e'_X E_i - E_i e'_Y, \end{aligned}$$

where $\bar{\delta}(M) = (\bar{\delta}(a_{ij}))$ for $M = (a_{ij})$. Since the bottom row of G is (0) or (00) and e'_X, e'_Y are upper triangular, (BRC) on $(\mathfrak{A}', \mathfrak{B}')$ follows, the proof is finished.

Lemma 4.3.3 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair with $\bar{\mathcal{T}}$ trivial, $|\bar{\mathcal{T}}| \geq 2$, and $\bar{\mathfrak{B}}_X$ having 4.2.1 (ii). Then the pair satisfies (BRC) with respect to the sets $(\mathcal{D}, \mathcal{U})$ and $(\mathcal{E}, \mathcal{W})$, where

$$\mathcal{D} = \{a_i, i \in \Lambda\} \cup \{b_j, j < n_0\}, \mathcal{U} = \{\underline{v}_i, i \in \Lambda\} \cup \{\bar{u}_j, j < n_0\}; \mathcal{E} = \{\bar{a}_\tau, 1 \leq \tau \leq s\} \cup \{\bar{b}\}.$$

Theorem 4.3.4 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair with $\bar{\mathcal{T}}$ trivial having at least two vertices, such that $\bar{\mathfrak{B}}_X$ satisfies Classification 4.2.1 (ii). Then there exists a reduction sequence:

$$(\bar{\mathfrak{A}}, \bar{\mathfrak{B}}) = (\bar{\mathfrak{A}}^0, \bar{\mathfrak{B}}^0), (\bar{\mathfrak{A}}^1, \bar{\mathfrak{B}}^1), \dots, (\bar{\mathfrak{A}}^\gamma, \bar{\mathfrak{B}}^\gamma), (\bar{\mathfrak{A}}^{\gamma+1}, \bar{\mathfrak{B}}^{\gamma+1}), \dots, (\bar{\mathfrak{A}}^{\kappa-1}, \bar{\mathfrak{B}}^{\kappa-1}), (\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa) \quad (4.3-1)$$

in the sense of Lemma 2.3.2. Where the κ -th pair possesses the minimal property that: if a row of Θ^κ contains some \bar{b} -class arrows of $\bar{\mathfrak{B}}^\kappa$, then the same row of F^κ contains one and only one nonzero entry which is a link in some G_κ^i given by an edge reduction.

(i) For $i = 0, 1, \dots, (\kappa - 2)$, the reduction from $\bar{\mathfrak{A}}^i$ to $\bar{\mathfrak{A}}^{i+1}$ is a composition of a series of reductions $\bar{\mathfrak{A}}^i = \bar{\mathfrak{A}}^{i,0}, \bar{\mathfrak{A}}^{i,1}, \dots, \bar{\mathfrak{A}}^{i,r_i}, \bar{\mathfrak{A}}^{i,r_i+1} = \bar{\mathfrak{A}}^{i+1}$, where.

- ① The reduction from $\bar{\mathfrak{A}}^{i,j}$ to $\bar{\mathfrak{A}}^{i,j+1}$ is a sequence of regularization for non-effective a, b -class arrows and finally an edge reduction of the form (0) for an effective a or b -class arrow, $0 \leq j < r_i - 1$. The reduction from $\bar{\mathfrak{A}}^{i,r_i-1}$ to $\bar{\mathfrak{A}}^{i,r_i}$ is a sequence of regularization for non-effective a, b -class arrows.
- ② The first arrow $a_1^i : X^i \mapsto Y^i$ of $\bar{\mathfrak{B}}^{i,r_i}$ is an effective a or b -class edge with $\delta(a_1^i) = 0$. Making an edge reduction $a_1^i \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we obtain $\bar{\mathfrak{A}}^{i,r_i+1} = \bar{\mathfrak{A}}^{i+1}$.

(ii) It is possible that there exist a minimal integer γ , and an index $1 \leq j \leq r_\gamma + 1$, such that the first arrow of $\bar{\mathfrak{B}}^{\gamma,j}$ locates outside the matrix block coming from \bar{b} , but the first arrow of $\bar{\mathfrak{B}}^{\gamma,j+1}$ locates at the first column of the block.

(iii) The reduction from $\bar{\mathfrak{A}}^{\kappa-1,0}$ to $\bar{\mathfrak{A}}^{\kappa-1,r_{\kappa-1}}$ is a composition of a series of reductions given by (i) ①. $\bar{\mathfrak{B}}^{\kappa-1,r_{\kappa-1}}$ is non-local. There are two possibilities:

- ① The first arrow $a_1^{\kappa-1}$ of $\bar{\mathfrak{B}}^{\kappa-1,r_{\kappa-1}}$ is an effective a or b -class solid edge with $\delta(a_1^{\kappa-1}) = 0$. Making an edge reduction $a_1^{\kappa-1} \mapsto (1)$ or $(0 \ 1)$, we obtain $\bar{\mathfrak{A}}^{\kappa-1,r_{\kappa-1}+1} = \bar{\mathfrak{A}}^\kappa$;
- ② The first arrow $a_1^{\kappa-1}$ is an effective b -class loop at the down-right corner of the matrix block splitting from \bar{b} with $\delta(a_1^{\kappa-1}) = 0$. Making a loop reduction $a_1^{\kappa-1} \mapsto W$, a trivial Weyr matrix over R^κ , we obtain $\bar{\mathfrak{A}}^{\kappa-1,r_{\kappa-1}+1} = \bar{\mathfrak{A}}^\kappa$.

Proof If the number of \bar{a} -class edges $s = 0$, after a series of regularization, we reach the unique effective loop with $\delta(\bar{b}) = 0$. Let $\bar{b} \mapsto W$, the final pair $(\bar{\mathfrak{A}}^1, \bar{\mathfrak{B}}^1)$ satisfies (iii) ② with $\kappa = 1$. Suppose $s > 0$, we make regularization for a_i, b_j before \bar{a}_1 , the corresponding $\underline{v}_i, \bar{u}_j = 0$. Thus $\delta(\bar{a}_1) = 0$ by Formula (4.2-7), if $\bar{a}_1 \mapsto (0)$, $\bar{\mathfrak{A}}^{0,1}$ of (i) ① follows. If $r_0 > 1$, repeating the procedure in (i) ①, we finally reach $\bar{\mathfrak{A}}^{0,r_0}$ with the first arrow a_1^0 and $\delta(a_1^0) = 0$. If a_1^0 is \bar{a} -class and $a_1^0 \mapsto (1), (01)$, we obtain (iii) ①; If a_1^0 is \bar{b} -class and $a_1^0 \mapsto W$, we obtain (iii) ② with $\kappa = 1$. Otherwise, if $a_1^0 \mapsto G^1, G^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 01 \\ 00 \end{pmatrix}$ in the case of (i) ②, we obtain the induced pair $(\bar{\mathfrak{A}}^1, \bar{\mathfrak{B}}^1)$.

Suppose we have reached $(\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i)$ for some $i < \kappa - 1$ given in (i), now continue the reductions up to the induced pair $(\bar{\mathfrak{A}}^{i+1}, \bar{\mathfrak{B}}^{i+1})$. $(\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i)$ satisfies (BRC) by Lemma 4.3.2-4.3.3. Suppose the first arrow of $\bar{\mathfrak{B}}^{i,0}$, $a_1^{i,0} = a_{\tau n^i l}$ or $b_{\tau' n^i l'}$, splits from a non-effective a_τ or $b_{\tau'}$ with n^i the lowest row index, and l or l' the column index of the splitting block, then $\delta(a_1^{i,0}) = \underline{v}_{\tau n^i l}$, or $\bar{u}_{\tau' n^i l'}$ by Formulae (4.2-7)-(4.2-8) and (4.1-8). Thus $a_1^{i,0} \mapsto \emptyset, \underline{v}_{\tau n^i l} = 0$ or $\bar{u}_{\tau' n^i l'} = 0$ by Remark 4.1.3 (ii). We continue regularization for the non-effective arrows inductively, and finally send an effective one to (0) by Lemme 4.1.3 (i), then obtain $\bar{\mathfrak{A}}^{i,1}$ at (i) ①. With the similar argument as above we reach $\bar{\mathfrak{A}}^{i,r_i}$ with the first arrow $\delta(a_1^i) = 0$ still by 4.1.3 (i). Let $a_1^i \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 01 \\ 00 \end{pmatrix}$, we obtain the $(i+1)$ -the pair.

$\bar{\mathfrak{A}}^{\kappa-1, r_{\kappa-1}}$ is not local, since $\bar{\mathfrak{A}}^{\kappa-1,0}$ is not by Condition (BRC). If $a_1^{\kappa-1}$ is an edge, then $a_1^{\kappa-1} \mapsto (1)$ or (01) gives the case (iii) ①; If $a_1^{\kappa-1}$ is a loop, then $a_1^{\kappa-1} \mapsto W$ gives the case (iii) ②. In both cases, the induced pair $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$ possesses the minimal property, the proof is finished.

Suppose $s(a_1^{i-1}) = X^{i-1}$ in the case of Theorem 4.3.4 (i) ②, the reduction on a_1^{i-1} gives $e_{X^{i-1}} \mapsto \begin{pmatrix} e_{Y^i} & \bar{w}^i \\ 0 & e_{X^i} \end{pmatrix}$ for $1 \leq i < \kappa$. Denote by \bar{W}_κ^i the split of \bar{w}^i in e_X^κ for $1 \leq i < \kappa$, which can be divided into $(\kappa - i)$ blocks: denote by n_i^κ the size of $e_{Y^i}^\kappa$, and by n_κ^κ that of $e_{X^\kappa}^\kappa$, which is 1 in the case of Theorem 4.3.4 (iii) ①; or as the same as that of W in the case of (iii) ②. Thus $\bar{W}_{\kappa j}^i$ has the size $n_i^\kappa \times n_{j+1}^\kappa$. Write $n^\kappa = \sum_{i=1}^\kappa n_i^\kappa$, the number of rows of e_X^κ .

$$e_X^\kappa = \begin{pmatrix} e_{Y^1}^\kappa & \bar{W}_{\kappa 1}^1 & \cdots & \cdots & \bar{W}_{\kappa, \kappa-1}^1 \\ & e_{Y^2}^\kappa & \cdots & \cdots & \bar{W}_{\kappa, \kappa-1}^2 \\ & & \cdots & \cdots & \\ & & & e_{Y^{\kappa-1}}^\kappa & \bar{W}_{\kappa, \kappa-1}^{\kappa-1} \\ & & & & e_{X^\kappa}^\kappa \end{pmatrix} \quad (4.3-2)$$

When we make a reduction, the dotted arrows appearing in J' , see Lemma 2.1.2, are said to be w -class. Where the dotted elements in \bar{W}_κ^i of Formula (4.3-2) for $1 \leq i < \kappa$, and their splits are said to be \bar{w} -class; those in $e_{Y^i}^\kappa, 1 \leq i < \kappa$, and $e_{X^\kappa}^\kappa$ are still said to be w -class. Moreover the dotted arrows first appearing at the ς -th step of reduction for $\varsigma > \kappa$ and their splits are also called w -class.

The following facts are already implied in the proof of Theorem 4.3.4.

Corollary 4.3.5 The elements in \bar{W}_κ^i of (4.3-2) for $1 \leq i < \kappa$ are dotted arrows of $\bar{\mathfrak{B}}^\kappa$.

4.4 Major pairs

We will show in this sub-section, that under some further assumption the one sided pairs given in Theorem 4.3.4 are not homogeneous.

Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair with $\bar{\mathfrak{B}}_X$ satisfying Classification 4.2.1 (ii), whose number of the \bar{a} -class arrows $s \geq 1$. According to the coefficients of the first two Formulae of (4.2-8), we define s linear combinations of the \bar{v} -class arrows in $\bar{\mathfrak{B}}$:

$$\hat{v}_\tau = \sum_j (\bar{\varepsilon}_{\tau j} - \sum_{\bar{a}_\tau \prec b_i \prec \bar{b}} \bar{\alpha}_i \bar{\varepsilon}_{i\tau j}) \bar{v}_j, \quad \tau = 1, \dots, s. \quad (4.4-1)$$

Fix any $1 \leq \tau \leq s$, making reductions according to Theorem 4.3.4 (iii) ① for $\kappa = 1$, such that $a_1^0 = \bar{a}_\tau \mapsto (1)$, we reach the induced pair $(\bar{\mathfrak{A}}^1, \bar{\mathfrak{B}}^1)$. Then we continue to do further reductions based on formulae (4.2-7)-(4.2-8) inductively for $\bar{a}_\eta \prec a_i, b_i \prec \bar{a}_{\eta+1}, \eta = \tau, \dots, s$, and $\bar{a}_s \prec a_i, b_i \prec \bar{b}$ by Remark 4.1.3 (ii):

$$a'_i \mapsto \emptyset, \underline{v}'_i + (1) \sum_j \epsilon_{i\tau j} \underline{u}'_j = 0, \quad b'_i \mapsto \emptyset, \bar{u}'_i + (1) \sum_j \epsilon_{i\tau j} \bar{v}'_j = 0; \quad (4.4-2)$$

on the other hand, $\bar{a}_\eta \mapsto \emptyset$ or (0) for $\tau < \eta \leq s$ according to $\delta(\bar{a}'_{\tau+\eta}) \neq 0$ or $= 0$. Replacing \bar{u}_i by \bar{v} -class arrows inductively by Remark 4.1.3 (iii), the second formula of (4.2-8) gives at the induced pair $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$:

$$\delta(\bar{b}') = \sum_{\bar{a}_\tau \prec b_i \prec \bar{b}} \bar{\alpha}_i \bar{u}'_i + (1) (\sum_j \bar{\epsilon}_{\tau j} \bar{v}'_j) = \sum_j (\bar{\epsilon}_{\tau j} - \sum_{\bar{a}_\tau \prec b_i \prec \bar{b}} \bar{\alpha}_i \epsilon_{i\tau j}) \bar{v}'_j = \hat{v}'_\tau. \quad (4.4-3)$$

Lemma 4.4.1 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair with $\bar{\mathcal{T}}$ trivial and $s \geq 1$, such that $\bar{\mathfrak{B}}_X$ satisfies Classification 4.2.1 (ii) and the c -class arrows have Formula (4.2-6). If there exists some $1 \leq \tau \leq s$, with $\hat{v}_\tau = 0$ in Formula (4.4-1), then $\bar{\mathfrak{B}}$ is wild and non-homogeneous.

Proof (i) Since $\bar{\mathfrak{B}}_X$ is minimal local with $R_X = k[x, \phi(x)^{-1}]$, set $\mathcal{L}' = k \setminus \{\text{roots of } \phi(x)\}$, there is an almost split conflation (e'_λ) for any $\lambda \in \mathcal{L}'$ in $R(\bar{\mathfrak{B}}_X)$. Let $\vartheta : R(\bar{\mathfrak{B}}_X) \rightarrow R(\bar{\mathfrak{B}})$ be the induced functor, if $\bar{\mathfrak{B}}$ is homogeneous, then there is a co-finite subset $\mathcal{L} \subseteq \mathcal{L}'$, and a set of almost split conflations $\{(e_\lambda) = \vartheta(e'_\lambda) : S'_\lambda \rightarrow E'_\lambda \rightarrow S'_\lambda \mid \lambda \in \mathcal{L}\}$ with $S_\lambda(\bar{b}) = (\lambda), E_\lambda(\bar{b}) = J_2(\lambda)$.

(ii) According to Formula (4.4-1)-(4.4-3), set $\bar{a}_\tau \mapsto (1)$, we have $\delta(\bar{b}') = 0$ at the induced pair $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$. Thus we are able to construct an object $L \in R(\bar{\mathfrak{B}})$ with $L_X = k, L_Y = k, L(\bar{a}_\tau) = (1), L(\bar{b}) = (\lambda)$ and others zero. Similar to the proof of Lemma 3.4.1, we obtain a contradiction, which shows that $\bar{\mathfrak{B}}$ is not homogeneous, the proof is finished.

Theorem 4.4.2 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair with R trivial and $s > 1$, such that $\bar{\mathfrak{B}}_X$ satisfies Lemma 4.2.1 (ii) and the c -class arrows have Formula (4.2-6). If the elements $\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_s\}$ defined in Formula (4.4-1) are linearly dependent, then $\bar{\mathfrak{B}}$ is wild and non-homogeneous.

Proof Without loss of generality, we may assume $\bar{\mathcal{T}} = \{X, Y\}$. Suppose there is a minimally linearly dependent subset of $\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_s\}$ with l vectors. Since the case of $l = 1$ has been proved by Lemma 4.4.1, we assume here that $l > 1$:

$$\begin{aligned} & \{\hat{v}_{\tau_1}, \hat{v}_{\tau_2}, \dots, \hat{v}_{\tau_l}\}, \quad \tau_1 < \tau_2 < \dots < \tau_l, \\ & \hat{v}_{\tau_1} = \beta_2 \hat{v}_{\tau_2} + \dots + \beta_r \hat{v}_{\tau_l}, \quad \beta_2, \dots, \beta_l \in k \setminus \{0\}. \end{aligned} \quad (4.4-5)$$

(i) Making reductions according to Theorem 4.3.4 (i) and (iii) ① for $\kappa = l$, such that $a_1^p = \bar{a}_{\tau_p, p} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $p = 1, \dots, l-1$, $a_1^l = \bar{a}_{\tau_l, l} \mapsto (1)$, we obtain an induced pair $(\bar{\mathfrak{A}}^l, \bar{\mathfrak{B}}^l)$. The reduced formal product $F^l + \bar{\Theta}^l$ looks like (with only \bar{a}, \bar{b}, c -class arrows):

0	0	1	...	$\bar{a}_{\tau_2, 1}$...	$\bar{a}_{\tau_3, 1}$...	$\bar{a}_{\tau_l, 1}$	$\bar{a}_{\tau_l+1, 1} \dots \bar{a}_{s1}$	$b_{11} \ b_{12} \ \dots \ b_{1l}$	$c_{11} \ \dots \ c_{t1}$
		0	0	1	...	$\bar{a}_{\tau_3, 2}$...	$\bar{a}_{\tau_l, 2}$	$\bar{a}_{\tau_l+1, 2} \dots \bar{a}_{s2}$	$b_{21} \ b_{22} \ \dots \ b_{2l}$	$c_{12} \ \dots \ c_{t2}$
				1	...	$\bar{a}_{\tau_l, 3}$	$\bar{a}_{\tau_l+1, 3} \dots \bar{a}_{s3}$	$b_{31} \ b_{32} \ \dots \ b_{3l}$	$c_{13} \ \dots \ c_{t3}$		
				0	...	1	$\bar{a}_{\tau_l+1, l} \dots \bar{a}_{sl}$	$b_{l1} \ b_{l2} \ \dots \ b_{ll}$	$c_{1l} \ \dots \ c_{tl}$		

(4.4-6)

We claim that the pair $(\bar{\mathfrak{A}}^l, \bar{\mathfrak{B}}^l)$ is local: $\bar{\mathfrak{A}}$ has two vertices, the dimension of $\vartheta^{0l}(F^l)$ in $R(\bar{\mathfrak{A}})$ equals $l+1$, and the number of links in F^l equals l , the assertion follows Corollary 2.3.4 (ii).

(ii) We make further reductions from $\bar{\mathfrak{B}}^l$ inductively for the \bar{p} -th row ordered by $\bar{p} = l, \dots, 2$ in the reduced formal product $\bar{\Theta}^l$. For $\bar{p} = l$, similar to Formulae (4.4-2)-(4.4-3): $a_{il} \mapsto \emptyset, i \in \Lambda, b_{ilq} \mapsto \emptyset, \bar{u}_{ilq} + (1) \sum_j \epsilon_{i\tau_l j} \bar{v}_{jlq} = 0, i < n_0; \bar{a}_{\eta l} \mapsto (0)$ or $\emptyset, \tau_l < \eta \leq s, \bar{b}_{lq} \mapsto \emptyset, \hat{v}_{lq} = 0, q = 1, \dots, l$. Next, $b_{ilq} \mapsto \emptyset$ for $i > n_0, 1 \leq q \leq l$ by Remark 4.1.3 (ii); and $c_{il} \mapsto (0)$ or \emptyset inductively. The dotted arrows \underline{v}_{ip} and \bar{u}_{ipq} for all i and $p < l$ are preserved by 4.1.3 (iv); note

that $\bar{v}_i : Y \mapsto X$, the size of the split of \bar{v}_i in $\bar{\Psi}^l$ is $l \times 1$, the \bar{v}_{ipq} for all i, q and $p < l$ are also preserved. The induced boc $\bar{\mathfrak{B}}^{l+1}$ follows.

(iii) Suppose we have reached an induced boc $\bar{\mathfrak{B}}^{2l-\bar{p}+1}$ for some $\bar{p} \leq l$, which satisfies the following two conditions. ① Denote the entrances of $F^{2l-\bar{p}}$ which do not belong to $\cup_{j=1}^l G_{2l-\bar{p}}^j$ by \bullet^0 coming from \bullet , one of the a, b, c -class solid arrows. Then $a_{ip}^0 = \emptyset, i \in \Lambda, \bar{a}_{ip}^0 = \emptyset$ or (0) , $b_{ip}^0 = \emptyset, i \neq n_0, \bar{b}_{ip}^0 = \emptyset, c_{ip} = \emptyset$ or (0) for any $p > \bar{p}$; ② The dotted arrows $\underline{v}_{ip}, \underline{u}_{ipq}; \bar{v}_{ipq}$ are preserved for all i, q and $p \leq \bar{p}$. Now we continue to make reductions for the solid arrows at the \bar{p} -th row of $\Theta^{2l-\bar{p}+1}$. We have $a_{i\bar{p}}^0 = \emptyset, i \in \Lambda$ and $b_{i\bar{p}q} = \emptyset, \bar{u}_{ipq} + \bar{a}_{\tau_p p}^0 \sum_j \varepsilon_{i\tau_p j} \bar{v}_{jpq} = 0, i < n_0$ inductively according to the assumptions ①-② and Remark 4.1.3 (ii); while $\bar{a}_{\eta, \bar{p}} \mapsto (0)$ or \emptyset for $\tau_{\bar{p}} < \eta \leq s$; $\delta(\bar{b}_{\bar{p}q}) = \hat{v}_{\bar{p}q}, \bar{b}_{\bar{p}q} \mapsto \emptyset, \hat{v}_{\bar{p}q} = 0, q = 1, \dots, l$. Moreover $b_{i\bar{p}q} \mapsto \emptyset, c_{i\bar{p}} \mapsto (0)$ or \emptyset similarly to the discussion in (ii). We finally reach the $(2l - \bar{p})$ -th pair with the assumptions ①-②. By induction, we obtain a pair $(\bar{\mathfrak{A}}^{2l-1}, \bar{\mathfrak{B}}^{2l-1})$.

(iv) The non-effective $a_{i1}, b_{j1q} \mapsto \emptyset$ for $i \in \Lambda, j < n_0, q = 1, \dots, l$, and $\bar{a}_{i1} \mapsto \emptyset$ or (0) for $\tau_1 < i \leq s$ at the first row of (4.4-6), we obtain an induced boc $\bar{\mathfrak{B}}^{2l}$. Formula (4.4-5) gives $\hat{v}_{\tau_1 1q} = \beta_2 \hat{v}_{\tau_2 2q} + \dots + \beta_r \hat{v}_{\tau_l lq} = 0$ thus $\delta^0(\bar{b}_{1q}) = 0$ for $q = 1, \dots, l$. Since $l \geq 2$, the boc $\bar{\mathfrak{B}}^{2l}$ is wild and non-homogeneous by Classification 4.2.1 (iii) or (iv). And hence so is $\bar{\mathfrak{B}}$.

Definition 4.4.3 A one-sided pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ with $\bar{\mathfrak{B}}_X$ satisfying Classification 4.2.1 (ii) is said to be a *major pair*, provided $\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_s\}$ in Formula (4.4-1) are linearly independent.

4.5 Further reductions

Throughout the subsection let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided major pair having at least two vertices, such that $\bar{\mathfrak{B}}_X$ satisfies Classification 4.2.1 (ii) and the c -class arrows satisfy Formula (4.2-6). Suppose $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$ is an induced pair given by Theorem 4.3.4 (iii). Let $(\bar{\mathfrak{A}}^\varsigma, \bar{\mathfrak{B}}^\varsigma)$ be an induced pair for some $\varsigma \geq \kappa$ by a series of reductions in the sense of Lemma 2.3.2. This subsection is devoted to discussing the reduction for $\bar{\mathfrak{B}}^\varsigma$ via calculating the differential of the first arrow.

Let $(\bar{\mathfrak{A}}^\varsigma, \bar{\mathfrak{B}}^\varsigma)$ be given above. Put a solid or dotted arrow in a square box; and a matrix block in a rectangle with four boundaries. The block G_ς^j for $1 \leq j \leq \kappa$ is defined in Formula (2.3-5), whose upper boundary is that of I_ς^j and denoted by m_ς^{j-1} , the lower one is that of F^ς , the left and right boundaries are given by the dotted lines l_ς^j and r_ς^j . Denote by $A_\varsigma^j, B_\varsigma^j, C_\varsigma^j \subset F^\varsigma + \Theta^\varsigma$, the sets of a, b, c -class entrances and solid arrows in the j -th block row, on the right hand side of I_ς^j , with the upper (resp. lower) boundary m_ς^{j-1} (resp. m_ς^j).

	I_ς^1	$A_\varsigma^1 \cup B_\varsigma^1$						$B_\varsigma^1 \cup C_\varsigma^1$	(4.5-1)	
		I_ς^2	$A_\varsigma^2 \cup B_\varsigma^2$							$B_\varsigma^2 \cup C_\varsigma^2$
			
		$I_\varsigma^{\gamma+1}$						$B_\varsigma^{\gamma+1} \cup C_\varsigma^{\gamma+1}$		
			
		$I_\varsigma^{\kappa-1}$						$B_\varsigma^{\kappa-1} \cup C_\varsigma^{\kappa-1}$		
		W_ς^κ						$B_\varsigma^\kappa \cup C_\varsigma^\kappa$		
		G_ς^1	G_ς^2	$G_\varsigma^{\gamma+1}$				$G_\varsigma^{\kappa-1}$	G_ς^κ	

In $A_\varsigma^j, B_\varsigma^j, C_\varsigma^j$, a solid arrow is denoted by $a_{\varsigma i, pq}^j, b_{\varsigma i, pq}^j, c_{\varsigma i, pq}^j$ splits from a_i, b_i, c_i respectively, an entry of F^ς by $\bullet_{\varsigma i, pq}^{j,0}$, where (p, q) is the index in the $n^\varsigma \times n_{t(a_i)}^\varsigma, n^\varsigma \times n^\varsigma, n^\varsigma \times n_{t(c_i)}^\varsigma$ -block

matrix respectively. If there is no confusion, we write \bullet_{ipq}^j and $\bullet_{ipq}^{j,0}$ for simple. For the sake of convenience, $\Phi_{\underline{m}^\varsigma}^m$ is partitioned also by the lines m, l, r as the same as in $F^\varsigma + \Theta^\varsigma$.

Remark From now on, we consider the pseudo reduced formal equation (4.1-7) at the ς -th step, in order to determine the linear relation on the dotted elements. Keep Remark 4.1.3 in mind. Since loop or edge reduction may add some w -class dotted arrows, but does not cause any new linear relations among the splits of dotted elements, we will describe the relationship of $\bar{u}, \bar{v}, \underline{u}, \underline{v}, \bar{w}, w$ -class elements during the regularization from $\bar{\mathfrak{B}}^\varsigma$ to $\bar{\mathfrak{B}}^{\varsigma+1}$. In the following three Lemmas, suppose the first arrow $a_1^\varsigma = \bullet_{\tau\bar{p}\bar{q}}^\iota$ of $\bar{\mathfrak{B}}^\varsigma$ belongs to $A_\varsigma^\iota \cup B_\varsigma^\iota \cup C_\varsigma^\iota$ in Picture (4.5-1), Write an element $\bullet_{ipq}^{j,0}$ of F^ς with $j \geq \iota$, $p > \bar{p}$, or $p = \bar{p}$ but $q < \bar{q}$; while write the solid arrow $\bullet_{i'p'q'}^{j'}$ with $j' \leq \iota$ and $p' < \bar{p}$, or $p' = \bar{p}$ but $q' \geq \bar{q}$.

Lemma 4.5.1 Let $(\mathfrak{A}^\varsigma, \mathfrak{B}^\varsigma)$ be an induced pair of $(\mathfrak{A}^\kappa, \mathfrak{B}^\kappa)$, the latter is given by Theorem 4.3.4 (iii) ②, and $\iota = \kappa$ with the first arrow $a_1^\varsigma = \bullet_{\tau\bar{p}\bar{q}}^\kappa \in B_\varsigma^\kappa \cup C_\varsigma^\kappa$, (see below the second thick line of Picture (4.5-1) for example). Assume that

- (i) Any $b_{ipq}^{\kappa,0} = \emptyset, i > n_0$, the corr. element \bar{u}_{ipq}^κ is replaced by a combination of some \bar{v} -class arrows in $\bar{\mathfrak{B}}^\varsigma$. While the dotted arrows $\bar{u}_{i'p'q'}^{j'}$ are preserved.
- (ii) If $c_{ipq}^{\kappa,0} = \emptyset$, there is a linear relation among some elements $\underline{v}_{i_1p_1q_1}^\kappa, h < i_1 \leq h + i, p_1 \geq p$ or $p_1 = p, q_1 < q$ and some \underline{u}, w -class. While all the dotted arrows $\underline{v}_{i'p'q'}^{j'}$ are preserved.

Then after a regularization, the induced pair $(\mathfrak{A}^{\varsigma+1}, \mathfrak{B}^{\varsigma+1})$ still satisfies (i)-(ii). In particular all the dotted arrows $\underline{v}_{i'p'q'}^{j'}, \bar{u}_{i'p'q'}^{j'}$ except $\underline{v}_{\tau\bar{p}\bar{q}}^\kappa, \bar{u}_{\tau\bar{p}\bar{q}}^\kappa$; and all the \bar{w}, \bar{v} -class arrows are preserved.

Proof The assumption is valid for $\varsigma = \kappa$ by Theorem 4.3.4 and Corollary 4.3.5.

- (i) If $a_1^\varsigma = b_{\tau\bar{p}\bar{q}}^\kappa, \tau > n_0$, then according to Formula (4.2-8), (4.1-8) and Remark 4.1.3 (i):

$$\delta(b_{\tau\bar{p}\bar{q}}^\kappa) = \bar{u}_{\tau\bar{p}\bar{q}}^\kappa + \sum_{n_0 < i < \tau} (\alpha_{\tau i}^0 \bar{u}_{i\bar{p}\bar{q}}^\kappa + \sum_q \alpha_{\tau i}^1 \bar{b}_{\bar{p}q}^{\kappa,0} \bar{u}_{iq\bar{q}}^\kappa) + \sum_{c_i < b_{\tau,q}} c_{i\bar{p}q}^{\kappa,0} (\sum_l \varepsilon'_{\tau il} \bar{v}_{lq\bar{q}}).$$

Since W is upper triangular, $\bar{p} \leq q$ in $\bar{b}_{\bar{p}q}^{\kappa,0}$. By the assumption (i), $\bar{u}_{\tau\bar{p}\bar{q}}^\kappa$ is a dotted arrow, thus $b_{\tau\bar{p}\bar{q}}^\kappa \mapsto \emptyset$, $\bar{u}_{\tau\bar{p}\bar{q}}^\kappa$ is replaced by a combination of some \bar{v} -class arrows, since $\bar{u}_{i\bar{p}\bar{q}}^\kappa, \bar{u}_{iq\bar{q}}^\kappa$ are already so by the assumption (i).

- (ii) If $a_1^\varsigma = c_{\tau\bar{p}\bar{q}}^\kappa$, then according to Formula (4.2-9), (4.1-8) and Remark 4.1.3 (i):

$$\delta(c_{\tau\bar{p}\bar{q}}^\kappa) = \sum_{h < i \leq h + \tau} (\gamma_{\tau i}^0 \underline{v}_{i\bar{p}\bar{q}}^\kappa + \sum_q \gamma_{\tau i}^1 \bar{b}_{\bar{p}q}^{\kappa,0} \underline{v}_{iq\bar{q}}^\kappa) + \sum_{i,q} c_{i\bar{p}q}^{\kappa,0} (\sum_l \xi_{\tau il} \underline{u}_{lq\bar{q}}) - \sum_{q < \bar{q}} c_{\tau\bar{p}q}^{\kappa,0} w_{q\bar{q}} + \sum_{p > \bar{p}} w_{\bar{p}p} c_{\tau\bar{p}q}^0.$$

In the case of $\delta(c_{\tau\bar{p}\bar{q}}^\kappa) \neq 0$, $c_{\tau\bar{p}\bar{q}}^\kappa \mapsto \emptyset$, which yields an additional linear relation among elements $\underline{v}_{i\bar{p}\bar{q}}, \underline{v}_{iq\bar{q}}, h < i \leq h + \tau, q \geq \bar{p}$, and some \underline{u}, w -class elements.

The required \underline{v}, \bar{u} -class and all the \bar{v}, \bar{w} -class dotted arrows are preserved, the pair $(\mathfrak{A}^{\varsigma+1}, \mathfrak{B}^{\varsigma+1})$ still satisfies assumption (i)-(ii), which finishes the proof.

Suppose in Formula (4.5-1), $I_\varsigma^j, j > \gamma$, intersects the p -th row of F^ς at the q_p^j -th column with $\bar{b}_{pq_p^j}^{j,0} = (1)$. Denote by $\bar{w}_{q_p^j}^j$ for any possible q the dotted element with row index q_p^j at e_x^ς . If $j > \iota$, or $j = \iota$ but $p > \bar{p}$, then $q_p^j > q_{\bar{p}}^\iota$, $\bar{w}_{q_p^j}^j$ is sitting below $\bar{w}_{q_{\bar{p}}^\iota}^\iota$, we refer to Picture (4.5-2).

Lemma 4.5.2 Let $(\bar{\mathfrak{A}}^\varsigma, \bar{\mathfrak{B}}^\varsigma)$ be an induced pair of $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$ with γ existing in Theorem 4.3.4 (ii). Suppose the first arrow $a_1^\varsigma = \bullet_{\tau\bar{p}\bar{q}}^\gamma \in B_\varsigma^\gamma \cup C_\varsigma^\gamma$ with $\gamma < \iota \leq \kappa$ in 4.3.4 (iii) ①, or $\gamma < \iota < \kappa$, in ②, (see between the two thick lines of Picture (4.5-1) for example). Assume that

- (i) Any $\bar{b}_{pq}^{j,0} = \emptyset$, the corr. $\bar{w}_{q_p^j}^j$ is replaced by a combination of some $\bar{w}_{lq}, l > q_p^j$ and w -class elements at $\bar{\mathfrak{B}}^\varsigma$. While the dotted arrows $\bar{w}_{p'q'}^{j'}$ with $p' \leq q_{\bar{p}}^\iota$ are preserved.
- (ii) Any $\bar{b}_{ipq}^{j,0} = \emptyset, i > n_0$, the corr. \bar{u}_{ipq}^j is replaced by a combination of some $\bar{u}_{i_1p_1q_1}^{j_1} \succ \bar{u}_{ipq}^j, n_0 < i_1 < i$, and \bar{v} -class elements in $\bar{\mathfrak{B}}^\varsigma$. While the dotted arrows $\bar{u}_{i'p'q'}^{j'}$ are preserved.

(iii) If $c_{ipq}^{j,0} = \emptyset$, there is a relation among elements $\underline{v}_{i_1 p_1 q_1}^{j_1}, h < i_1 \leq h + i$, and \underline{u}, w -class. While the dotted arrow $\underline{v}_{i' p' q'}^{j'}$, $i' \in \Lambda$, are preserved.

Then after a regularization, the induced pair $(\mathfrak{A}^{\varsigma+1}, \mathfrak{B}^{\varsigma+1})$ still satisfies (i)-(iii). In particular all the dotted arrows $\underline{v}_{i' p' q'}^{j'}$, $i' \in \Lambda$; $\bar{u}_{i' p' q'}^{j'}$ except $\bar{u}_{\tau \bar{p} \bar{q}}^j$; $\bar{w}_{p' q', p' < q_p^j}$; and \bar{v} -class are preserved.

Proof The assumption (i)-(iii) are valid, if the block of a_1^ς has the bottom and right boundaries $(m_\varsigma^\kappa, r_\varsigma^\kappa)$ in the case of Theorem 4.3.4 (iii) ①; or $(m_\varsigma^{\kappa-1}, r_\varsigma^{\kappa-1})$ in (iii) ② by Lemma 4.5.1.

(i) If $a_1^\varsigma = \bar{b}_{\bar{p} \bar{q}}^\iota$, by the notation $\bar{b}_{\bar{p} \bar{q}}^{\iota,0} = (1)$, and by the assumption (i) $\bar{b}_{\bar{p} \bar{q}}^{\iota,0} = \emptyset, q < \bar{q}$,

$$\delta(\bar{b}_{\bar{p} \bar{q}}^\iota) = (1) \bar{w}_{q_p^\iota \bar{q}} + \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p} p} \bar{b}_{p \bar{q}}^{j,0}, \quad w_{\bar{p} p} \text{ belongs to } \bar{w} \text{ or } w \text{ class.}$$

Since $\bar{w}_{q_p^\iota \bar{q}}$ is a dotted arrow, $\bar{b}_{\bar{p} \bar{q}}^\iota \mapsto \emptyset$ at the induced pair $(\mathfrak{A}^{\varsigma+1}, \mathfrak{B}^{\varsigma+1})$. As an example, we refer to \bar{W}_ς^3 and \bar{W}_ς^4 in the picture (4.5-2) below.

(ii) If $a_1^\varsigma = b_{\tau \bar{p} \bar{q}}^\iota$, $\tau > n_1$, then $\bar{b}_{\bar{p} \bar{q}}^{\iota,0} = (1)$, $\bar{b}_{\bar{p} \bar{q}}^\iota = \emptyset, \forall q > q_p^\iota$ by (i) above; and for some $\iota_1 \leq \iota$:

$$\delta(b_{\tau \bar{p} \bar{q}}^\iota) = \bar{u}_{\tau \bar{p} \bar{q}} + \sum_{n_0 < i < \tau} (\alpha_{\tau i}^0 \bar{u}_{i \bar{p} \bar{q}}^\iota + \alpha_{\tau i}^1 \bar{b}_{\bar{p} q_p^\iota}^{\iota_1} \bar{u}_{i q_p^\iota \bar{q}}^{\iota_1}) + \sum_{c_i < b_{\tau, q}} c_{i \bar{p} q}^0 (\sum_l \varepsilon'_{\tau i l} \bar{v}_{l q \bar{q}}), \quad b_{\tau \bar{p} \bar{q}}^\iota \mapsto \emptyset,$$

since $\bar{u}_{\tau \bar{p} \bar{q}}$ is a dotted arrow by the assumption (ii), which is replaced by some \bar{u}, \bar{v} -class elements.

(iii) If $a_1^\varsigma = c_{\tau \bar{p} \bar{q}}^\iota$, in case of $c_{\tau \bar{p} \bar{q}}^\iota \mapsto \emptyset$, a relation among $\underline{u}, \underline{v}, i > h$, w -class elements is added:

$$\delta(c_{\tau \bar{p} \bar{q}}^\iota) = \sum_{h < i \leq h + \tau} (\gamma_{\tau i}^0 \underline{v}_{i \bar{p} \bar{q}}^\iota + \gamma_{\tau i}^1 \bar{b}_{\bar{p} q_p^\iota}^{\iota_1} \underline{v}_{i q_p^\iota \bar{q}}^{\iota_1}) + \sum_{i, q} c_{i \bar{p} q}^{\iota,0} (\sum_l \xi_{\tau i l} \underline{u}_{l q \bar{q}}) - \sum_{q < \bar{q}} c_{\tau \bar{p} q}^{\iota,0} w_{q \bar{q}} + \sum_{p > \bar{p}} w_{\bar{p} p} c_{\tau \bar{p} p}^{j,0}.$$

The required $\underline{v}, \bar{u}, \bar{w}$ -class and all the \bar{v} -class dotted arrows are preserved, the pair $(\mathfrak{A}^{\varsigma+1}, \mathfrak{B}^{\varsigma+1})$ still satisfies assumption (i)-(iii), which finishes the proof.

Suppose in the picture (4.5-1), I_ς^j comes from an effective \bar{a}_{ij} for $j \leq \gamma$ if γ exists, otherwise $j \leq \kappa$ in Theorem 4.3.4 (iii) ①, or $j < \kappa$ in (iii) ②, which intersects the p -th row at the q_p^j -th column in the $n_X^\varsigma \times n_{t(a_{ij})}^\varsigma$ -block coming from \bar{a}_{ij} partitioned by $\bar{\mathcal{T}}$ with $\bar{a}_{ij, p q_p^j}^{j,0} = (1)$. Denote by $\hat{v}_{ij, q_p^j}^j$ splitting from \hat{v}_{ij} for any possible q , whose row index is q_p^j in the $n_{t(a_{ij})}^\varsigma \times n_X^\varsigma$ -block. We write \hat{V}_ς^j at the (n_0, h_i) -th block partitioned by $\bar{\mathcal{T}}$ in $\bar{\Psi}_{\bar{u}}^\varsigma$, and refer to Picture (4.5-2).

Lemma 4.5.3 Let $(\bar{\mathfrak{A}}^\varsigma, \bar{\mathfrak{B}}^\varsigma)$ be an induced boc of $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$. Suppose the first arrow $a_1^\varsigma = \bullet_{\tau \bar{p} \bar{q}} \in A_\varsigma^\iota \cup B_\varsigma^\iota \cup C_\varsigma^\iota$ with $\iota \leq \gamma$ if γ exists; otherwise $j \leq \kappa$ in Theorem 4.3.4 (iii) ①, or $j < \kappa$ in (iii) ②, (see above the first thick line of Picture (4.5-1) for example). Assume that

(i) Any $a_{ipq}^{j,0} = \emptyset, i \in \Lambda$, the corr. \underline{v}_{ipq}^j is replaced by a combination of some \underline{u} -class elements at $\bar{\mathfrak{B}}^\varsigma$. While all the dotted arrows $\underline{v}_{i' p' q'}^{j'}$ are preserved.

(ii) If an effective $\bar{a}_{ipq}^{j,0} = \emptyset$, there is a linear relation among $\underline{u}, \bar{w}, w$ -class elements at $\bar{\mathfrak{B}}^\varsigma$. While all the dotted arrows $\underline{v}_{i' p' q'}^{j'}$, $i' \in \Lambda$, are preserved.

(iii) Any $b_{ipq}^{j,0} = \emptyset, i < n_0$, the corresponding \bar{u}_{ipq}^j is replaced by a combination of some \bar{v} -class elements at $\bar{\mathfrak{B}}^\varsigma$. While all the dotted arrows $\bar{u}_{i' p' q'}^{j'}$ are preserved.

(iv) Any $\bar{b}_{pq}^{j,0} = \emptyset, \bar{v}_{ij q_p^j}$ corr. to $\bar{a}_{ij p q_p^j}^{j,0}$ is replaced by a combination of some \bar{v} -class elements below and some \bar{w}, w -class at $\bar{\mathfrak{B}}^\varsigma$. While the dotted arrows $\bar{v}_{i' p' q'}^{j'}$ with $p' \leq q_p^j$ are preserved.

(v) Any $b_{ipq}^{j,0} = \emptyset, i > n_0$, the corr. element \bar{u}_{ipq}^j is replaced by a combination of some $\bar{u}_{i_1 p q}^j$, $n_0 < i_1 < i$, and \bar{v} -elements at $\bar{\mathfrak{B}}^\varsigma$. While all the dotted arrows $\bar{u}_{i' p' q'}^{j'}$ are preserved.

(vi) If $c_{ipq}^{j,0} = \emptyset$, there is a linear relation among some elements $\underline{v}_{i_1 p q}^j, h < i_1 < h + \tau$, and $\underline{u}, w, \bar{w}$ -class at $\bar{\mathfrak{B}}^\varsigma$. While all the dotted arrows $\underline{v}_{i' p' q'}^{j'}$, $i' \in \Lambda$, are preserved.

Then after a regularization, the induced pair $(\mathfrak{A}^{\varsigma+1}, \mathfrak{B}^{\varsigma+1})$ still satisfies (i)-(vi). In particular, all the dotted arrows $\underline{v}_{i' p' q'}^{j'}$, $i' \in \Lambda$, except $\underline{v}_{\tau \bar{p} \bar{q}}^j$; $\bar{u}_{i' p' q'}^{j'}$ except $\bar{u}_{\tau \bar{p} \bar{q}}^j$; and $\hat{v}_{p' q', p' < q_p^j}$ are preserved.

Proof The assumption (i)-(vi) are valid, if a_1^ζ has the bottom and right boundaries $(m_\zeta^\gamma, r_\zeta^\gamma)$ when γ exists by Lemma 4.5.2, otherwise $(m_\zeta^\kappa, r_\zeta^\kappa)$ in Theorem 4.3.4 (iii) ①, or $(m_\zeta^{\kappa-1}, r_\zeta^{\kappa-1})$ in (iii) ② by Lemma 4.5.1.

(i) If $a_1^\zeta = a_{\tau\bar{p}\bar{q}}^\zeta$ with $\tau \in \Lambda$, since $\underline{v}_{\tau\bar{p}\bar{q}}$ is a dotted arrow by the assumption (i):

$$\delta(a_{\tau\bar{p}\bar{q}}^\zeta) = \underline{v}_{\tau\bar{p}\bar{q}}^\zeta + \sum_{i,q} \bar{a}_{i\bar{p}q}^{\zeta,0} (\sum_l \epsilon_{\tau il} \underline{u}_{lq\bar{q}}) \implies a_{\tau\bar{p}\bar{q}}^\zeta \mapsto \emptyset, \quad \underline{v}_{\tau\bar{p}\bar{q}}^\zeta = - \sum_{i,q} \bar{a}_{i\bar{p}q}^{\zeta,0} (\sum_l \epsilon_{\tau il} \underline{u}_{lq\bar{q}}),$$

(ii) If $a_1^\zeta = \bar{a}_{\tau p q}$ effective, by substituting $\underline{v}_{i'\bar{p}\bar{q}}^\zeta$ given in (i):

$$\begin{aligned} \delta(\bar{a}_{\tau p q}^\zeta) &= \sum_{\bar{a}_{\tau^\iota} \prec a_{i'} \prec \bar{a}_\tau} \bar{\beta}_{\tau i'} \underline{v}_{i'\bar{p}\bar{q}}^\zeta + \sum_{i,q} \bar{a}_{i\bar{p}q}^{\zeta,0} (\sum_l \bar{\epsilon}_{\tau il} \underline{u}_{lq\bar{q}}) - \sum_{q < \bar{q}} \bar{a}_{\tau\bar{p}q}^{\zeta,0} w_{q\bar{q}} + \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p}p} \bar{a}_{\tau\bar{p}q}^{j,0} \\ &= \sum_i \bar{a}_{i\bar{p}q}^{\zeta,0} (\sum_{l,q} (\bar{\epsilon}_{\tau il} - \sum_{\bar{a}_{\tau^\iota} \prec a_{i'} \prec \bar{a}_\tau} \bar{\beta}_{\tau i'} \epsilon_{i' il}) \underline{u}_{lq\bar{q}}) - \sum_{q < \bar{q}} \bar{a}_{\tau\bar{p}q}^{\zeta,0} w_{q\bar{q}} + \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p}p} \bar{a}_{\tau\bar{p}q}^{j,0}. \end{aligned}$$

If $\delta(\bar{a}_{\tau p q}^\zeta) \neq 0$, $\bar{a}_{\tau p q}^\zeta \mapsto \emptyset$ yields a linear relation among some \underline{u} , w and \bar{w} -class elements.

(iii) If $a_1^\zeta = b_{\tau\bar{p}\bar{q}}^\zeta$, $\tau < n_0$, since $\bar{u}_{\tau\bar{p}\bar{q}}^\zeta$ is a dotted arrow by the assumption (iii):

$$\delta(b_{\tau\bar{p}\bar{q}}^\zeta) = \bar{u}_{\tau\bar{p}\bar{q}}^\zeta + \sum_{i,q} \bar{a}_{i\bar{p}q}^{\zeta,0} (\sum_l \epsilon_{\tau il} \bar{v}_{lq\bar{q}}) \implies b_{\tau\bar{p}\bar{q}}^\zeta \mapsto \emptyset, \quad \bar{u}_{\tau\bar{p}\bar{q}}^\zeta = - \sum_{i,q} \bar{a}_{i\bar{p}q}^{\zeta,0} (\sum_l \epsilon_{\tau il} \bar{v}_{lq\bar{q}}).$$

(iv) If $a_1^\zeta = \bar{b}_{\bar{p}\bar{q}}^\zeta$ effective, by substituting $\bar{u}_{i'\bar{p}\bar{q}}^\zeta$ given in (iii), and Formula (4.4-1):

$$\begin{aligned} \delta(\bar{b}_{\bar{p}\bar{q}}^\zeta) &= \sum_{i' < n_0} \bar{\alpha}_{i'} \bar{u}_{i'\bar{p}\bar{q}}^\zeta + \sum_{i,q} \bar{a}_{i\bar{p}q}^{\zeta,0} (\sum_l \bar{\epsilon}_{il} \bar{v}_{lq\bar{q}}) - \sum_{q < \bar{q}} \bar{b}_{\bar{p}q}^{\zeta,0} w_{q\bar{q}} + \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p}p} \bar{b}_{\bar{p}\bar{q}}^{j,0} \\ &= \sum_{i,q} \bar{a}_{i\bar{p}q}^{\zeta,0} (\sum_l (\bar{\epsilon}_{il} - \sum_{\bar{a}_{\tau^\iota} \prec \bar{a}_{i'} \prec b_i \prec \bar{b}} \bar{\alpha}_{i'} \epsilon_{i' il}) \bar{v}_{lq\bar{q}}) - \sum_{q < \bar{q}} \bar{b}_{\bar{p}q}^{\zeta,0} w_{q\bar{q}} + \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p}p} \bar{b}_{\bar{p}\bar{q}}^{j,0} \\ &= \hat{v}_{\tau^\iota, q\bar{p}\bar{q}} + \sum_{\bar{a}_{i\bar{p}q}^{\zeta,0} \succ \bar{a}_{\tau^\iota, p\bar{q}}^{\zeta,0}; q} \bar{a}_{i\bar{p}q}^{\zeta,0} \hat{v}_{iq\bar{q}} - \sum_{q < \bar{q}} \bar{b}_{\bar{p}q}^{\zeta,0} w_{q\bar{q}} + \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p}p} \bar{b}_{\bar{p}\bar{q}}^{j,0}, \end{aligned}$$

since $\bar{a}_{\tau^\iota, p\bar{q}}^{\zeta,0} = 1$, $\hat{v}_{\tau^\iota, q\bar{p}\bar{q}}$ is a dotted arrow by the assumption (iv), $\bar{b}_{\bar{p}\bar{q}}^\zeta \mapsto \emptyset$, $\hat{v}_{\tau^\iota, q\bar{p}\bar{q}}$ is replaced by some \bar{w} , w -class; and some \hat{v} -class elements below $q\bar{p}$. See $\hat{V}_\zeta^1, \hat{V}_\zeta^2$ in Picture (4.5-2).

(v) If $a_1^\zeta = b_{\tau\bar{p}\bar{q}}^\zeta$, $\tau > n_0$, since $\bar{b}_{\bar{p}q}^0 = \emptyset$ for all possible q by (iv) above:

$$\delta(b_{\tau\bar{p}\bar{q}}^\zeta) = \bar{u}_{\tau\bar{p}\bar{q}}^\zeta + \sum_{i \neq n_0, i < \tau} \alpha_{\tau i}^0 \bar{u}_{i\bar{p}\bar{q}}^\zeta + \sum_{i,q} \bar{a}_{i\bar{p}q}^{\zeta,0} (\sum_l \epsilon_{\tau il} \bar{v}_{lq\bar{q}}) + \sum_{c_i \prec b_\tau; q} \bar{c}_{i\bar{p}q}^{\zeta,0} (\sum_l \epsilon'_{\tau il} \bar{v}_{lq\bar{q}}).$$

Since $\bar{u}_{\tau\bar{p}\bar{q}}^\zeta$ is a dotted arrow by the assumption (v), $b_{\tau\bar{p}\bar{q}}^\zeta \mapsto \emptyset$, $\bar{u}_{\tau\bar{p}\bar{q}}^\zeta$ is replaced by some $\bar{u}_{i\bar{p}\bar{q}}$, $n_0 < i < \tau$, and \bar{v} -elements by a replacement in (iii).

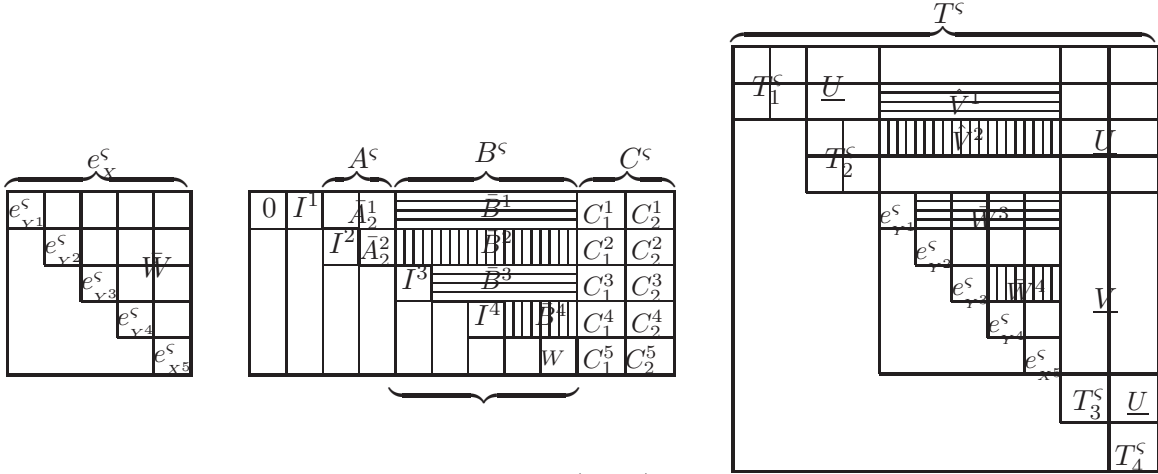
(vi) If $a_1^\zeta = c_{\tau\bar{p}\bar{q}}^\zeta$, since $\bar{b}_{\bar{p}q}^0 = \emptyset$ for all possible q , by (iv):

$$\begin{aligned} \delta(c_{\tau\bar{p}\bar{q}}^\zeta) &= \sum_{i \leq h+\tau} \gamma_{\tau i}^0 \underline{v}_{i\bar{p}\bar{q}}^\zeta + \sum_{i,q} \bar{a}_{i\bar{p}q}^{\zeta,0} (\sum_l \zeta_{\tau il} \underline{u}_{lq\bar{q}}) \\ &\quad + \sum_{i < \tau; q} \bar{c}_{i\bar{p}q}^{\zeta,0} (\sum_l \xi_{\tau il} \underline{u}_{lq\bar{q}}) - \sum_{q < \bar{q}} c_{\tau\bar{p}q}^{\zeta,0} w_{q\bar{q}} + \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p}p} c_{\tau\bar{p}\bar{q}}^{j,0}. \end{aligned}$$

Suppose $\delta(c_{\tau\bar{p}\bar{q}}^\zeta) \neq 0$, $c_{\tau\bar{p}\bar{q}}^\zeta \mapsto \emptyset$ causes a linear relation among $\underline{v}_{i\bar{p}\bar{q}}^\zeta$, $h < i \leq h+\tau$, and \underline{u} , w , \bar{w} -class elements by a replacement in (i).

The required \underline{v} , \bar{u} , \bar{w} , \bar{v} -class dotted arrows are preserved, the pair $(\bar{\mathfrak{A}}^{\zeta+1}, \bar{\mathfrak{B}}^{\zeta+1})$ still satisfies assumption (i)-(vi), which finishes the proof.

The following picture shows a pseudo formal equation $\bar{\Theta}^\zeta$ of $(\bar{\mathfrak{A}}^\zeta, \bar{\mathfrak{B}}^\zeta)$ for $\kappa = 5, \gamma = 2$ in the case of Theorem 4.3.4 (iii) ② with only effective arrows. From this we are able to see the correspondence of $(\bar{B}_\zeta^1, \hat{V}_\zeta^1), (\bar{B}_\zeta^2, \hat{V}_\zeta^2)$ and $(\bar{B}_\zeta^3, \bar{W}_\zeta^3), (\bar{B}_\zeta^4, \bar{W}_\zeta^4)$:



Picture (4.5-2)

4.6 Regularization for non-effective a class and all the b class arrows

Let a one-sided pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ and an induced pair $(\bar{\mathfrak{A}}^s, \bar{\mathfrak{B}}^s)$ be given in the beginning of section 4.5. It is clear by Lemma 4.5.1-4.5.3, that all the b -class and non-effective a -class solid arrows go to \emptyset by regularization. Now assume that we meet a local pair $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$ induced after the ς -th step still in the sense of Lemma 2.3.2, which has an induced pair in Classification 3.3.5 (iv):

$$(\bar{\mathfrak{A}}'_{(\lambda_0, \lambda_1, \dots, \lambda_{l-1})}, \bar{\mathfrak{B}}'_{(\lambda_0, \lambda_1, \dots, \lambda_{l-1})}) \quad (4.6-1)$$

Denote by $(\bar{\mathfrak{A}}^s, \bar{\mathfrak{B}}^s)$ an induced pair of the pair (4.6-1) after a series of regularization with the first loop a_1^s and $\delta(a_1^s) = 0$. Make a loop mutation, we obtain an induced pair $(\bar{\mathfrak{A}}^{s+1}, \bar{\mathfrak{B}}^{s+1})$ with $R^{s+1} = k[x]$. We claim in particular that W appearing in Theorem 4.3.4 (iii) ② must be trivial, since $\bar{\mathfrak{B}}^{\kappa-1, r_{\kappa-1}}$ is not local, but x appears only in a local bocs in the case of MW5.

Suppose $\bar{\mathfrak{B}}^{s+1}$ satisfies Formulae (3.3-3)-(3.3-4), an induced pair $(\bar{\mathfrak{A}}^t, \bar{\mathfrak{B}}^t)$ is in the case of MW5, then the non-effective a -class and all the b -class solid arrows are regularized during the reductions. In particular, the parameter x and the first arrow a_1^t of $\bar{\mathfrak{B}}^t$ belongs to \bar{a} or c -class. In fact, the discussion of 4.5.1-4.5.3 is still valid if we describe the linear relationship over the fractional field $k(x)$ of the polynomial ring $k[x]$, or over the field $k(x, x_1)$ of two indeterminants instead of the base field k .

Theorem 4.6.1 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one sided pair having at least two vertices, such that the induced local bocs $\bar{\mathfrak{B}}_X$ satisfies Classification 4.3.1 (ii), where the pair is major and the c -class arrows satisfy Formula (4.2-6). If $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ has an induced pair $(\bar{\mathfrak{A}}^t, \bar{\mathfrak{B}}^t)$ in the case of MW5, then the parameter x and the first arrow a_1^t must split from some \bar{a} or c -class arrows.

Finally, let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair having at least two vertices, such that $\bar{\mathfrak{B}}_X$ satisfies Classification 4.2.1 (i). Then $\bar{\mathfrak{B}}$ has only a, b -class solid arrows, where b_1, \dots, b_n are all non-effective; and $a_i, i \in \Lambda$ satisfying the first formula of (4.2-5) are non-effective, while $\bar{a}_i = a_{h_i}, i = 1, \dots, s$, satisfying the second one are effective. If there is an induced pair $(\bar{\mathfrak{A}}^t, \bar{\mathfrak{B}}^t)$ in the case of MW5, we have the following observation.

(i) Let $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$ be an induced pair of $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$, with $\bar{\mathcal{T}}'$ being trivial and the reduced formal product $\bar{\Theta}'$ given by Formula (4.1-6). Similarly to condition 4.3.1, let $\mathcal{D} = \{d_1, \dots, d_r\}$ be a set of solid arrows, $\mathcal{E} = \{e_1, \dots, e_s\}$ that of edges, such that $\mathcal{D} \cup \mathcal{E}$ form the lowest row of $\bar{\Theta}'$. Let $\mathcal{U} = \{u_1, \dots, u_r\}$ be a set of dotted arrows, with $\mathcal{W} = \mathcal{V}' \setminus \mathcal{U}$, such that $\bar{\delta}(d_i)$ and $\bar{\delta}(e_i)$ satisfy the formulae in 4.3.1 (ii), we obtain $(\text{BRC})'$. Then after a reduction given by Lemma 4.3.2, the induced pair still satisfies $(\text{BRC})'$; and the original pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ satisfies $(\text{BRC})'$ similar to Lemma 4.3.3, but the proofs are much easier than those.

(ii) For constructing a reduction sequence of $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ up to $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$, we need only the part (i) and (iii) ① of Theorem 4.3.4.

(iii) For the further reductions, we need only Theorem 4.5.3 (i)-(iii), then reach an induced pair $(\bar{\mathfrak{A}}^s, \bar{\mathfrak{B}}^s)$, where all the b -class, non-effective a -class arrows are regularized step by step.

Corollary 4.6.2 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one sided pair having at least two vertices, such that the induced local boc $\bar{\mathfrak{B}}_X$ satisfies Classification 4.3.1 (i). If $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ has an induced pair $(\bar{\mathfrak{A}}^t, \bar{\mathfrak{B}}^t)$ satisfying MW5, then the parameter x and the first arrow a_1^t must split from some \bar{a} -class arrows.

5. Non-homogeneity of wild bipartite problems

This section is devoted to proving the main Theorem 0.3. The way to do so is based on proving the non-homogeneous property for a wild bipartite matrix bi-module problem having BDCC condition in the case of MW5.

5.1 An inspiring example

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ with R trivial be a bipartite matrix bi-module problem having BDCC condition of representation wild type. Suppose $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ is an induced bi-module problem satisfying MW5. We classify the position of the first arrow a'_1 in the formal product Θ' of $(\mathfrak{A}', \mathfrak{B}')$:

$$H' + \Theta' = H' + \sum_{i=1}^{n'} a'_i * A'_i. \quad (5.1-1)$$

Denote by (p, q') the leading position of A'_1 over \mathcal{T}' , which locates in the (p, q) -th leading block of some base matrix of \mathcal{M}_1 partitioned by \mathcal{T} , with $q = q_Z$ a main block column over Z .

Classification 5.1.1 Let the pairs $(\mathfrak{A}, \mathfrak{B}), (\mathfrak{A}', \mathfrak{B}')$ be given above with \mathfrak{B}' satisfying MW5. Then there are two possible position relations between p and the row indices of the links of H' in Formula (5.1-1):

case (I) $p <$ the row indices of all the links in the (p, q) -block of H' ;

case (II) $p \geq$ some row index of at least one link in the (p, q) -block of H' .

It is clear that there is no link above the (p, q) -th block, since \mathfrak{A}' is already local.

Lemma 5.1.2 Let p_x be the row index of x in H' , then $p_x > p$ in Classification 5.1.1.

Proof. Since x appears before the first arrow a'_1 of \mathfrak{B}' , $p_x \geq p$ by the ordering of the reductions. If $p_x = p$, then the parameter x locates at the left side of a'_1 in $H' + \Theta'$, $\delta(a'_1)$ contains only the terms of the form $\alpha x v, \alpha \in k$, which contradicts to the assumption that \mathfrak{B}' is in the case of MW5. Thus $p_x > p$, the proof is finished.

Example 5.1.3 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair of matrix bi-module problem associated to the algebra defined in Example 1.4.5. There is a reduction sequence $\mathfrak{A} = \mathfrak{A}^0, \mathfrak{A}^1, \mathfrak{A}^2, \mathfrak{A}^3$ given in Examples 2.4.5, such that the corresponding boc \mathfrak{B}^3 of \mathfrak{A}^3 is strongly homogeneous in the case of MW5 described in 3.1.5 (iii). In order to prove that $(\mathfrak{A}, \mathfrak{B})$ is not homogeneous, we must find another way different from the proof of MW1-MW4. More precisely, we will reconstruct a new reduction

sequence based on the matrix \tilde{M} over $k[x]$ with the size vector $\underline{\tilde{m}} = (2, 2, 2, 2, 2, 3, 3, 3, 3, 3)$:

$$\tilde{M} = \left(\begin{array}{c|c|c|c|c} \begin{array}{ccc} 0 & & \\ & 0 & \\ & & 0 \end{array} & \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} & \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} & \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} & \begin{array}{ccc} 0 & & \\ & 0 & \\ & & 0 \end{array} \\ \hline \begin{array}{ccc} & & \\ & 0 & \\ & & 0 \end{array} & \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} & \begin{array}{ccc} 0 & & \\ & 0 & \\ & & 0 \end{array} & \begin{array}{ccc} 0 & & \\ & 0 & \\ & & 0 \end{array} & \begin{array}{ccc} 0 & 0 & 0 \\ 0 & x & 0 \end{array} \\ \hline \begin{array}{ccc} & & \\ & & 0 \end{array} & \begin{array}{ccc} 0 & & \\ & 0 & \\ & & 0 \end{array} & \begin{array}{ccc} 0 & & \\ & 0 & \\ & & 0 \end{array} & \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} & \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \\ \hline \begin{array}{ccc} & & \\ & & \\ & & 0 \end{array} & \begin{array}{ccc} & & \\ & & \\ & & 0 \end{array} & \begin{array}{ccc} 0 & & \\ & 0 & \\ & & 0 \end{array} & \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} & \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \\ \hline \begin{array}{ccc} & & \\ & & \\ & & 0 \end{array} & \begin{array}{ccc} & & \\ & & \\ & & 0 \end{array} & \begin{array}{ccc} & & \\ & & \\ & & 0 \end{array} & \begin{array}{ccc} & & \\ & & \\ & & 0 \end{array} & \begin{array}{ccc} & & \\ & & \\ & & 0 \end{array} \end{array} \right).$$

Corresponding to steps (i)-(iii) of 2.4.5, there is a reduction sequence $\mathfrak{A} = \tilde{\mathfrak{A}}^0, \tilde{\mathfrak{A}}^1, \tilde{\mathfrak{A}}^2, \tilde{\mathfrak{A}}^3$, where the reductions from $\tilde{\mathfrak{A}}^0$ to $\tilde{\mathfrak{A}}^1$ is given by $a \mapsto (01)$ in the sense of Lemma 2.3.2. Thus b splits into b_1, b_2 in $\tilde{\mathfrak{B}}^1$, and set $b_1 \mapsto (0), b_2 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ from $\tilde{\mathfrak{A}}^1$ to $\tilde{\mathfrak{A}}^2$. $\tilde{\mathfrak{A}}^3$ is obtained from $\tilde{\mathfrak{A}}^2$ by an edge reduction going to (0) , a loop mutation, then a series of regularization:

$$\tilde{H}^3 = \begin{pmatrix} 0 & 1_X & 0 \\ 0 & 0 & 1_X \end{pmatrix} * A + \begin{pmatrix} 0 & 0 & 1_X \\ 0 & 0 & 0 \end{pmatrix} * B + \begin{pmatrix} \emptyset & \emptyset & \emptyset \\ 0 & x1_X & \emptyset \end{pmatrix} * C.$$

The $(1, 5)$ -th block partitioned under \mathcal{T} in the formal equation of $(\tilde{\mathfrak{A}}^3, \tilde{\mathfrak{B}}^3)$ is of the form:

$$\begin{pmatrix} e & v \\ 0 & e \end{pmatrix} \begin{pmatrix} d_{10} & d_{11} & d_{12} \\ d_{20} & d_{21} & d_{22} \end{pmatrix} + \begin{pmatrix} u_{11}^1 & u_{12}^1 \\ u_{21}^1 & u_{22}^1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} u_{11}^2 & u_{12}^2 \\ xv & u_{22}^2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_{00}^2 & v_{01}^2 & v_{02}^2 \\ v_{10}^2 & v_{11}^2 & v_{12}^2 \\ 0 & vx & v_{11}^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{00}^1 & v_{01}^1 & v_{02}^1 \\ v_{10}^1 & v_{11}^1 & v_{12}^1 \\ v_{20}^1 & v_{21}^1 & v_{22}^1 \end{pmatrix} + \begin{pmatrix} d_{10} & d_{11} & d_{12} \\ d_{20} & d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} s_{00} & s_{01} & s_{02} \\ 0 & e & v \\ 0 & 0 & e \end{pmatrix}$$

with $e = e_X, s_{00} = e_Y$. Then the differentials of the solid arrows of $\tilde{\mathfrak{B}}^3$ can be read off:

$$d_{20} : X \mapsto Y, \delta(d_{20}) = 0; \quad d_{21} : X \mapsto X, \delta(d_{21}) = xv - vx - d_{20}s_{01}.$$

And $\bar{\delta}(d_{22}) = u_{21}^1 - v_{11}^2 - d_{20}s_{02} - d_{21}v, \bar{\delta}(d_{10}) = -v_{10}^2 - v_{20}^1 + vd_{20}, \bar{\delta}(d_{11}) = u_{11}^2 - v_{11}^2 - v_{21}^1 - d_{10}s_{01}, \bar{\delta}(d_{12}) = u_{11}^1 + u_{11}^2 - v_{12}^1 - v_{22}^1$, where $\bar{\delta}$ is respect to d_{20}, d_{21} . It is clear that the boc $\tilde{\mathfrak{B}}^3$ is in the case of Proposition 3.4.5, since for $d_{20} \mapsto (1)$, the solid loops $d_{21}, d_{22}, d_{10}, d_{11}, d_{12}$ will be regularized, because $s_{01}, u_{21}^1, v_{10}^2, u_{11}^2, u_{11}^1$ are pairwise different dotted arrows. Therefore $(\mathfrak{A}, \mathfrak{B})$ is not homogeneous.

Motivated by Example 5.1.3, we consider the general cases. Since the example satisfies Case (I) of Classification 5.1.1, we start from Case (I) in the subsection 5.1-5.3.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a bipartite matrix bi-module problem having RDCC condition. Let \mathfrak{A}' be an induced matrix bi-module problem with R' trivial. Let $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$ be the induced functor, $M = \vartheta^{0i}(H'(k)) = \sum_j M_j * A_j \in R(\mathfrak{A})$. Suppose that the size vector of M is $l \times \underline{n}$ over \mathcal{T} . Let $\mathbf{q} = \mathbf{q}_Z \in T_2$ for some $Z \in \mathcal{T}$. Define a size vector $l \times \underline{n}$ over \mathcal{T} , and construct a representation $\tilde{M} = \sum_j \tilde{M}_j * A_j \in R(\mathfrak{A})$ with 0 a zero column as follows:

$$\tilde{n}_j = \begin{cases} n_j, & \text{if } j \notin Z, \\ n_j + 1, & \text{if } j \in Z. \end{cases} \quad \tilde{M}_j = \begin{cases} M_j, & \text{if } A_j 1_Z = 0, \\ (0 M_j), & \text{if } A_j 1_Z = A_j. \end{cases} \quad (5.1-2)$$

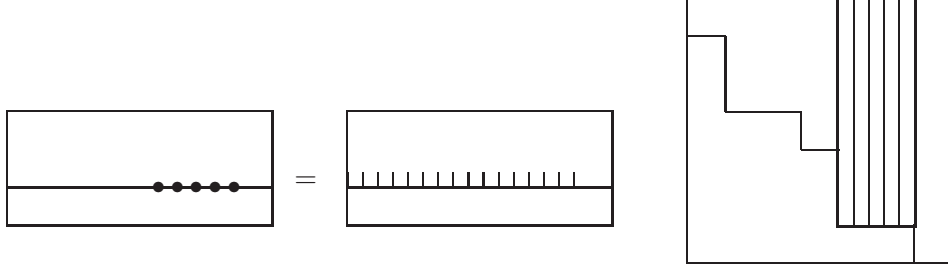
Denote by $(p, q+1)$ the leading position of M , and suppose that $q+1$ is the index of the first column of the \mathbf{q} -th block-column, denote by \tilde{q} the index of the added column in the \mathbf{q}_Z -th block-column of \tilde{M} . Consider the defining system \mathbb{E} of $\mathcal{K}'_0 \oplus \mathcal{K}'_1$ given by Formula (2.4-4), and a matrix equation $\tilde{\mathbb{E}}$:

$$\mathbb{E} : \Phi_l^1 M \equiv_{\prec(p, q+1)} M \Phi_{\underline{n}}^2, \quad \tilde{\mathbb{E}} : \Phi_l^1 \tilde{M} \equiv_{\prec(p, \tilde{q})} \tilde{M} \Phi_{\underline{n}}^2. \quad (5.1-3)$$

Using the right hand side of \mathbb{E} and $\tilde{\mathbb{E}}$, one defines two new matrix equations respectively:

$$\mathbb{E}_\tau : 0 \equiv_{\prec(p, q+1)} M \Phi_{\underline{n}}^2, \quad \tilde{\mathbb{E}}_\tau : 0 \equiv_{\prec(p, \tilde{q})} \tilde{M} \Phi_{\underline{n}}^2. \quad (5.1-4)$$

Since \mathfrak{A} has BDCC condition, the main block-column in $\Phi_{\underline{n}}^2$ determined by $Z \in \mathcal{T}$ can be written as $\Phi_{\underline{n}, Z}^2 = (\Phi_1^2, \dots, \Phi_{n_i}^2)^T$, such that either $\Phi_l^2 = 0$ or $\Phi_l^2 = (z_{pq}^l) \neq 0$, where z_{pq}^l are algebraically independent variables over k . Denote the pairwise different non-zero blocks by $\Phi_{l_1}^2, \dots, \Phi_{l_u}^2$.



Taken any integer $p' \geq p$ and $1 \leq j \leq n_Z$, the $(p', q+j)$ -th entry of the right side of \mathbb{E}_τ is:

$$\sum_{\Phi_l \neq 0} \sum_{q'} \alpha_{p'q'}^l z_{q', q+j}^l. \quad (5.1-5)$$

It is easy to see that $z_{q', q+j_1}^l$ and $z_{q', q+j_2}^l$ have the same coefficient $\alpha_{p', q'}^l$, the (p', q') -th entry of $H(k)$, in Formula (5.1-5). In the picture above, $n_Z = 5, p' = p$, the five equations locating at five circles have the same coefficients of each variables.

Remark 5.1.4 (i) The $(p, q+j_1)$ -the equation is a linear combination of the previous equations in \mathbb{E}_τ , if and only if so is the $(p, q+j_2)$ -th equation by Formula (5.1-5). Similarly, we have the same result in $\tilde{\mathbb{E}}_\tau$.

(ii) The equations in the system $\tilde{\mathbb{E}}$ (resp. $\tilde{\mathbb{E}}_\tau$) and those in \mathbb{E} (resp. \mathbb{E}_τ) are the same everywhere except at the \tilde{q} -column, since the entries of \tilde{M} in the \tilde{q}' -th, the first column of the q' -th block column with $q' \sim q$ under the partition \mathcal{T} , are all zero.

(iii) If there exists some $1 \leq j \leq n_Z$, such that the $(p, q+j)$ -th equation is a linear combination of the previous equations in \mathbb{E}_τ , then so is the (p, \tilde{q}) -th equation in $\tilde{\mathbb{E}}$, by (i)-(ii).

(iv) If the $(p, q+j)$ -th equation is a linear combination of the previous equations in \mathbb{E} , then so is it in \mathbb{E}_τ , since the variables in Φ_l and $\Phi_{\underline{n}}$ are algebraically independent.

5.2 Bordered trivial matrices in the bipartite case

This sub-section is devoted to constructing a reduction sequence based on a given sequence and a bordered matrix, which generalizes Example 5.1.3.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a bipartite matrix bi-module problem having RDCC condition. Let $\mathfrak{A}^s = (R^s, \mathcal{K}^s, \mathcal{M}^s, H^s)$ be an induced problem with R^s trivial and local. Then Corollary 2.3.5 gives a unique reduction sequence with each reduction being in the sense of Lemma 2.3.2:

$$\mathfrak{A} = \mathfrak{A}^0, \mathfrak{A}^1, \dots, \mathfrak{A}^i, \mathfrak{A}^{i+1}, \dots, \mathfrak{A}^s, \quad (*)$$

Let $\vartheta^{0s} : R(\mathfrak{A}^s) \rightarrow R(\mathfrak{A})$ be the induced functor, $M = \vartheta^{0s}(H^s(k)) = \sum_j M_j * A_i \in R(\mathfrak{A})$ with the size vector $l \times \underline{n}$, and $M_j = G_s^j(k)$ given by Formula (2.3-5). Define a size vector $l \times \tilde{n}$ and a representation $\tilde{M} = \sum_j \tilde{M}_j * A_j \in R(\mathfrak{A})$ given by Formula (5.1-2).

Theorem 5.2.1 There exists a unique reduction sequence based on the sequence (*):

$$\mathfrak{A} = \tilde{\mathfrak{A}}^0, \tilde{\mathfrak{A}}^1, \dots, \tilde{\mathfrak{A}}^i, \tilde{\mathfrak{A}}^{i+1}, \dots, \tilde{\mathfrak{A}}^s \quad (*)$$

where $\tilde{\mathfrak{A}}^i = (\tilde{R}^i, \tilde{\mathcal{K}}^i, \tilde{\mathcal{M}}^i, \tilde{H}^i)$, the reduction from $\tilde{\mathfrak{A}}^i$ to $\tilde{\mathfrak{A}}^{i+1}$ is a reduction or a composition of two reductions in the sense of Lemma 2.3.2. Moreover, $\tilde{\mathcal{T}}^s$ has two vertices, and

$$\tilde{\vartheta}^{0s}(\tilde{H}^s(k)) = \tilde{M}.$$

Proof We may assume that $l \times \underline{n}$ is sincere over \mathcal{T} . Otherwise, after a suitable deletion, we are able to obtain an induced problem \mathfrak{A}' , which is still bipartite having BDCC condition, such that M is a representation of \mathfrak{A}' of size vector $l' \times \underline{n}'$, which is sincere over \mathcal{T}' .

We will construct a sequence (*) inductively. The original term in the sequence is $\tilde{\mathfrak{A}}^0 = \mathfrak{A}^0$. Suppose that we have constructed a sequence $\tilde{\mathfrak{A}}^0, \tilde{\mathfrak{A}}^1, \dots, \tilde{\mathfrak{A}}^i$ for some $0 \leq i < s$, and $\tilde{\vartheta}^{0i} : R(\tilde{\mathfrak{A}}^i) \mapsto R(\tilde{\mathfrak{A}}^0)$ is the induced functor, such that there exists a representation

$$\tilde{M}^i = \tilde{H}_{\tilde{m}^i}^i(k) + \sum_{j=1}^{n^i} \tilde{M}_j^i * \tilde{A}_j^i \in R(\tilde{\mathfrak{A}}^i), \quad \text{with } \vartheta^{0i}(\tilde{M}^i) \simeq \tilde{M} \in R(\tilde{\mathfrak{A}}^0). \quad (5.2-1)$$

Write $M^i = \vartheta^{is}(H^s(k)) = H_{\underline{m}^i}^i(k) + \sum_{j=1}^{n^i} M_j^i * A_j^i \in R(\mathfrak{A}^i)$, where $M_1^i = G_s^{i+1}(k)$ by Corollary 2.4.5, denoted $G_s^{i+1}(k)$ by B for simplicity. Denote the first column of the \mathfrak{q} -th block-column in the formal product Θ^i , which B belongs to, by β . Now we are constructing $\tilde{\mathfrak{A}}^{i+1}$.

Case 1 $\tilde{\mathcal{T}}^i = \mathcal{T}^i$ and $\tilde{\mathfrak{A}}^i = \mathfrak{A}^i$.

1.1 $B \cap \beta$ is empty. Then $\tilde{G}^{i+1} = G^{i+1}$, $\tilde{H}^{i+1} = H^{i+1}$ and $\tilde{\mathfrak{A}}^{i+1} = \mathfrak{A}^{i+1}$.

Before giving the following cases, we claim that if $B \cap \beta$ is non-empty, B thus G^{i+1} can not be Weyr matrices. Otherwise, the first arrow a_1^i of \mathfrak{B}^i will be a loop. Since $\tilde{\mathcal{T}}^i = \mathcal{T}^i$, \tilde{a}_1^i will also be a loop and hence the numbers of rows and columns of \tilde{B} are the same. When the matrix B is enlarged by one column, then also enlarged by one row, a contradiction to the construction of \tilde{M} . So B is either a regularization block \emptyset or an edge reduction block $\begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix}$.

1.2 $B \cap \beta$ is non-empty, and $B = \emptyset$ is a zero block. Then $\tilde{B} = (\emptyset \ B)$ with \emptyset a zero column, $\tilde{H}^{i+1} = H^{i+1}$ and $\tilde{\mathfrak{A}}^{i+1} = \mathfrak{A}^{i+1}$ by a regularization.

1.3 $B \cap \beta$ is non-empty, $B = \begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix}$ and $r < \text{the number of columns of } B$. Then $\tilde{B} = (0 \ B)$ with 0 a zero column, $\tilde{H}^{i+1} = H^{i+1}$ and $\tilde{\mathfrak{A}}^{i+1} = \mathfrak{A}^{i+1}$ by an edge reduction.

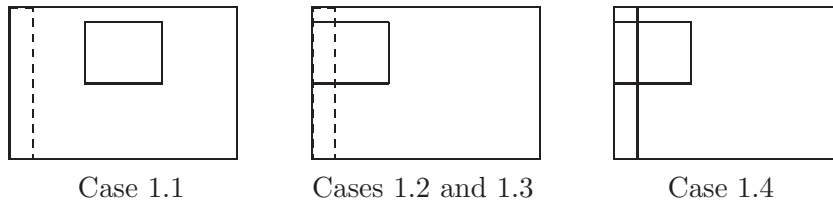
1.4 $B \cap \beta$ is non-empty, $B = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$. Then $\tilde{B} = (0 \ B)$ with 0 a zero column. Recall the formula (2.3-5) and Corollary 2.3.5, we define

$$\tilde{G}^{i+1} = \begin{cases} \begin{pmatrix} 0 & 1_{Z_2} \end{pmatrix}, & \text{if } G^{i+1} = (1_{Z_2}); \\ \begin{pmatrix} 0 & 1_{Z_2} \\ 0 & 0 \end{pmatrix}, & \text{if } G^{i+1} = \begin{pmatrix} 1_{Z_2} \\ 0 \end{pmatrix}, \end{cases} \quad \tilde{H}^{i+1} = \sum_{X \in \mathcal{T}^i} I_X * H_X^i + \tilde{G}^{i+1} * A_1^i.$$

Then $\tilde{\mathfrak{A}}^{i+1}$ is induced from $\tilde{\mathfrak{A}}^i$ by an edge reduction in the sense of Lemma 2.3.2.

We stress, that after the edge reduction 1.4, $\tilde{\mathcal{T}}^{i+1} = \mathcal{T}^{i+1} \cup \{Y\}$ with Y an equivalent class consisting of the indices of the added columns in the formal product $\tilde{\Theta}^{i+1}$ of the pair $(\tilde{\mathfrak{A}}^{i+1}, \tilde{\mathfrak{B}}^{i+1})$, and $(\tilde{\mathfrak{A}}^{i+1}, \tilde{\mathfrak{B}}^{i+1}) \neq (\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1})$ from this stage.

We show the above B as a small block in the corr. leading block partitioned by \mathcal{T} :



Case 2. $\tilde{\mathcal{T}}^i = \mathcal{T}^i \cup \{Y\}$.

2.1 $B \cap \beta$ is empty. Then $\tilde{B} = B$, $\tilde{G}^{i+1} = G^{i+1}$, and $\tilde{H}^{i+1} = \sum_{\tilde{X} \in \tilde{\mathcal{T}}^i} \tilde{I}_{\tilde{X}} * \tilde{H}_{\tilde{X}}^i + \tilde{G}^{i+1} * \tilde{A}_1$.

If $B \cap \beta$ is non-empty. Denote by \tilde{a}_0^i and \tilde{a}_1^i the first and the second solid arrows of $\tilde{\mathfrak{B}}^i$, which locate at (p^i, \tilde{q}_0^i) and $(p^i, \tilde{q}_0^i + 1)$ in the formal product $\tilde{\Theta}^i$ respectively.

2.2 $B \cap \beta$ is non-empty, and there exists some $1 \leq j \leq n_Z^i$, such that the $(p^i, q^i + j)$ -th equation is a linear combination of previous equations in \mathbb{E}_τ^i . Then $\delta(\tilde{a}_0^i) = 0$ by Remark 5.1.4 (iii) and Corollary 2.4.4. We make two reductions: the first is an edge reduction given by $\tilde{a}_0^i \mapsto (0)$; the second one for \tilde{a}_1^i is as the same as that for a_1^i by 5.1.4 (ii), then we obtain an induced problem $\tilde{\mathfrak{A}}^{i+1}$ with $\tilde{B} = (0 B)$, where 0 is a zero column.

2.3 $B \cap \beta$ is non-empty, and for all $1 \leq j \leq m_Z^i$, the $(p^i, q^i + j)$ -th equation is not a linear combination of previous equations in \mathbb{E}_τ^i . Thus $\delta(\tilde{a}_0^i) \neq 0$ still by 5.1.4 (iii) and 2.4.4. And $(p^i, q^i + j)$ -th equation neither is in \mathbb{E}^i by 5.1.4 (iv), $\delta(a_1^i) \neq 0$ again by 2.4.4. Then we make two regularization $\tilde{a}_0^i \mapsto \emptyset$, $\tilde{a}_1^i \mapsto \emptyset$, and $\tilde{B} = (\emptyset B)$ with \emptyset being a zero column.

In the cases 2.2-2.3, we define $\tilde{G}^{i+1,0} = (0)$ or \emptyset , $\tilde{G}^{i+1,1} = G^{i+1}$, and set $\tilde{H}^{i+1} = \sum_{\tilde{X} \in \tilde{\mathcal{T}}^i} \tilde{I}_{\tilde{X}} * \tilde{H}_{\tilde{X}}^i + \tilde{G}^{i+1,0} * \tilde{A}_0^i + \tilde{G}^{i+1,1} * \tilde{A}_1^i$.

By summary up all the cases, we obtain an induced pair $(\tilde{\mathfrak{A}}^{i+1}, \tilde{\mathfrak{B}}^{i+1})$ and a representation \tilde{M}^{i+1} satisfying Formula (5.2-1). The theorem follows by induction.

Corollary 5.2.2 With the notations as in Theorem 5.2.1. The main diagonal block $\tilde{e}_Z^i, Z \in \mathcal{T}$, of $\tilde{\mathcal{K}}_0^i \oplus \tilde{\mathcal{K}}_1^i$ has the form with $m = n_Z^i$:

$$\begin{pmatrix} s_{00} & s_{01} & s_{02} & \cdots & s_{0m} \\ & s_{11} & s_{12} & \cdots & s_{1m} \\ & & s_{22} & \cdots & s_{2m} \\ & & & \ddots & \vdots \\ & & & & s_{mm} \end{pmatrix}.$$

Where $s_{01}, s_{02}, \dots, s_{0m}$ are dotted arrows of $\tilde{\mathfrak{B}}^i$.

Proof By the construction of \tilde{H}^i , the added “0-column” can be only 0 or \emptyset . Therefore except s_{00} , the elements at the 0-th row: $s_{01}, s_{02}, \dots, s_{0m}$ do not appear in the defining equation system of $\tilde{\mathfrak{A}}^i$, and thus they are free. The proof is finished.

5.3 Bordered matrices with a parameter x in the bipartite case

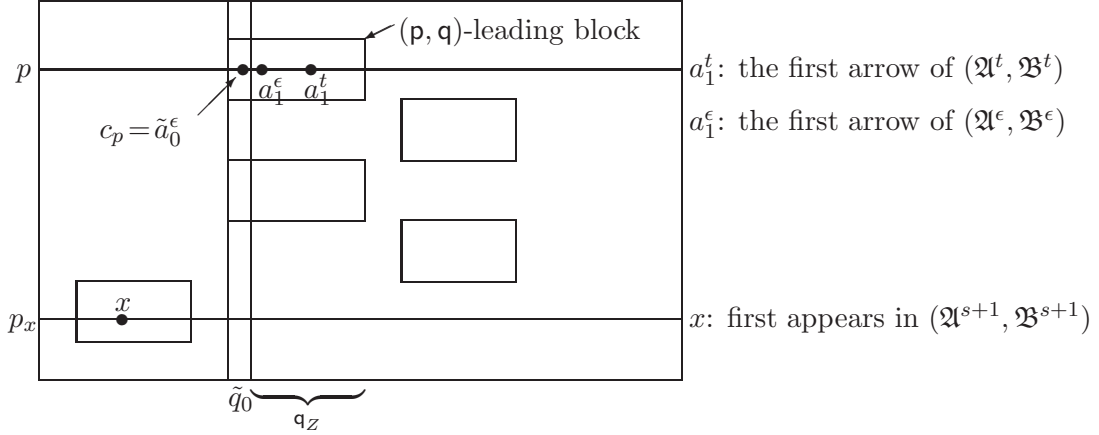
This sub-section is devoted to proving that the bipartite pair with an induced minimal wild pair of MW5 and satisfying Classification 5.1.1 (I) is not homogeneous.

Suppose we have a reduction sequence:

$$\mathfrak{A} = \mathfrak{A}^0, \mathfrak{A}^1, \dots, \mathfrak{A}^s, \mathfrak{A}^{s+1}, \dots, \mathfrak{A}^\epsilon, \dots, \mathfrak{A}^\epsilon, \dots, \mathfrak{A}^t = \mathfrak{A}' \quad (*')$$

where the reduction from \mathfrak{A}^i to \mathfrak{A}^{i+1} is given by Lemma 2.3.2 for $1 \leq i = 1 < s$, and \mathfrak{A}^s is local; the reduction from \mathfrak{A}^s to \mathfrak{A}^{s+1} is a loop mutation, then we obtain a parameter x at the (p_x, q_x) -position and $R^{s+1} = k[x]$; the reduction from \mathfrak{A}^i to \mathfrak{A}^{i+1} is a regularization for $s < i < t$. The pair $(\mathfrak{A}^t, \mathfrak{B}^t)$ is in the case of MW5 and Classification 5.1.1 (I).

Note that the set T^i of integers and its partition \mathcal{T}^i of \mathfrak{A}^i are all the same for $i = s, \dots, t$, we may write \underline{n}^i uniformly by \underline{n} ; t^i by \bar{t} . Suppose the first arrow a_1^t of \mathfrak{B}^t locates at (p, q') -position in the formal product Θ^t with $q' = q + j$ for some $1 \leq j \leq n_Z^t$; the first arrow a_1^ϵ of \mathfrak{B}^ϵ locates at $(p, q + 1)$ in Θ^ϵ , where $q + 1$ is the index of the first column in the q -th block-column. The picture below shows the position of the first solid arrows in the formal products Θ^i of $(\mathfrak{A}^i, \mathfrak{B}^i)$ for $i = s, \epsilon, t$ (when we ignore the added \tilde{q}_0 -th column):



For any $s < i \leq t$, assume that $R^i = k[x, \phi^i(x)^{-1}]$ in \mathfrak{A}^i , $H^i \in \mathbb{M}_{ti}(k[x, \phi^i(x)^{-1}])$. Then H^i are all in the same form but with different $\phi^i(x)$. Since $k(x)$ is a $k[x, \phi^i(x)^{-1}]$ -bi-module,

$$H^i \otimes_{R^i} 1_{k(x)} \in \mathbb{M}_{ti}(k[x, \phi^i(x)^{-1}]) \otimes_{R^i} k(x) \simeq \mathbb{M}_{ti}(k(x)).$$

Remark 5.1.4 (i) is still valid when we consider the equations over $k(x)$ instead of k in \mathbb{E}_τ and $\tilde{\mathbb{E}}_\tau$. Define a variable matrix $\Phi_{\tilde{n}}^{(\bullet)}$ of size \tilde{n} with the \tilde{q} -th column as the same as that of $\Phi_{\tilde{n}}^2$ and others zero. In the systems below, see Formula (5.1-4), we only need to consider the second one:

$$\tilde{\mathbb{E}}_\tau : 0 \equiv \tilde{M}\Phi_{\tilde{n}}^2; \quad \tilde{\mathbb{E}}_\tau^{(\bullet)} : 0 \equiv \tilde{M}\Phi_{\tilde{n}}^{(\bullet)}, \quad (5.3-1)$$

Suppose x locates at the p_x -th row of the (p_x, q_Z) -th main block partitioned under \mathcal{T} , see the picture above for the case of $Z' \neq Z$; and the example 5.1.3 for $Z' = Z$. Thus the equation system $\tilde{\mathbb{E}}_\tau^{(>p_x)}$ consisting of the equations below the p_x -th row is over the base field k . Denote by $\mathcal{K}_\tau^{(>p_x)} \subset \mathbb{M}_{\tilde{t} \times \tilde{t}}(k)$ the solution space of $\tilde{\mathbb{E}}_\tau^{(>p_x)}$. Since $\tilde{\mathcal{K}}_1^s \oplus \tilde{\mathcal{K}}_1^s$ is local and upper triangular, one base matrix E of $\mathcal{K}_\tau^{(>p_x-1)}$ with entries all zero, except 1_Y at the position (\tilde{q}, \tilde{q}) , $\{E\}$ forms the basis of $\mathcal{K}_{\tau,0}^{(>p_x-1)}$; and other base matrix $V_j, j \geq 1$, with non-zero entries above the \tilde{q} -row at the \tilde{q} -th column, form a basis of $\mathcal{K}_{\tau,1}^{(>p_x-1)}$.

Suppose we have solved the equations $\tilde{\mathbb{E}}_\tau^{(>h)}$ for some $p < h \leq p_x$, and obtained $R_\tau^{(>h)} = k[x, \prod_{\eta=p}^{h-1} d^\eta(x)^{-1}] \times k1_Y$; and a quasi-free module $\mathcal{K}_{\tau,1}^{(>h)}$ with a quasi-basis $\{U_1, \dots, U_\kappa\}$ over $R_\tau^{(>h)} \otimes_k R_\tau^{(>h)}$. Consider the formal product and the equation system, see Theorem 2.4.1:

$$\Pi_\tau^{(>h)} = \sum_{\zeta=1}^\kappa u_\zeta * U_\zeta, \quad \tilde{\mathbb{E}}_\tau^{(>h)} : 0 \equiv H(k(x))\Pi_\tau^{(>h)}. \quad (5.3-2)$$

Where the h -th equation of $\tilde{\mathbb{E}}_\tau^{(>h)}$ is $\sum_{\zeta=1}^\kappa f_\zeta(x)u_\zeta, f_\zeta(x) \in R_\tau^{(>h)}$. There are two possibilities:

(i) $f_\zeta(x) = 0$ for $\zeta = 1, \dots, \kappa$, then $\mathcal{K}_{\tau,1}^{(>h-1)} = \mathcal{K}_{\tau,1}^{(>h)}$, set $d^h(x) = 1$ and the quasi-basis of $\mathcal{K}_{\tau,1}^{(>h)}$ are preserved in $\mathcal{K}_{\tau,1}^{(>h-1)}$.

(ii) There exists some $f_\zeta(x) \neq 0$, without loss of generality we may assume that $f_\kappa(x) \neq 0$. Choose a new basis of $\text{Hom}_{k(x)}(\mathcal{K}_{\tau,1}^{(>h)} \otimes_{R^{(>h)}} k(x), k(x))$ at the first line of Formula (5.3-3) below, we have the corresponding base change of $\mathcal{K}_{\tau,1}^{(>h)}$ over $R_\tau^{(>h)}$ at the second line:

$$\begin{cases} u'_\zeta = u_\zeta, \\ U'_\zeta = U_\zeta - f_\zeta(x)/f_\kappa(x)U_\kappa; \end{cases} \quad \text{for } 1 \leq \zeta < \kappa; \quad \begin{cases} u'_\kappa = \sum_{\zeta=1}^\kappa f_\zeta(x)u_\zeta, \\ U'_\kappa = 1/f_\kappa(x)U_\kappa. \end{cases} \quad (5.3-3)$$

Where $u_\kappa = 0$ is the solution of the (h, \tilde{q}) -th equation in the system (5.3-2), thus $\mathcal{K}_{\tau,1}^{(>h-1)}$ possesses the quasi-free-basis $\{U_\zeta \mid \zeta = 1, \dots, \kappa - 1\}$. Let $d^h(x) \in k[x]$ be the numerator of $f_\kappa(x)$, and $R_\tau^{(>h-1)} = k[x, \prod_{\eta=1}^h d^\eta(x)] \times k1_Y$.

By induction, we finally reach the equation system $\tilde{\mathbb{E}}^{(>p-1)}$ with the solution space $\mathcal{K}_{\tau,1}^{(>p-1)}$ and a polynomial $d^p(x)$. Consider the pair $(\mathfrak{A}^i, \mathfrak{B}^i)$ for some $s+1 \leq i \leq \epsilon$, suppose $R^i = k[x, \phi^i(x)^{-1}]$, and the row index of the first arrow is $p^i, p_x \leq p^i \leq p$, in the formal product Θ^i . Define

$$\tilde{\phi}^i(x) = \phi^i(x) \Pi_{\eta=p_x}^{p^i} d^p(x) \in k[x], \quad \text{in particular} \quad \tilde{\phi}^t(x) = \phi^t(x) \Pi_{\eta=p_x}^p d^p(x). \quad (5.3-4)$$

Denote $M^i = H^i(k[x, \tilde{\phi}^i(x)^{-1}]) = \sum_j M_j^i * A_j$ having the size vector $l \times \underline{n}$ over \mathcal{T} . Then we are able to construct a size vector $l \times \tilde{n}$ over \mathcal{T} ; and a matrix $\tilde{M}^i = \sum_j \tilde{M}_j^i * A_j$ by Formula (5.1-2). Write the matrix equations for $i \geq s$:

$$\begin{aligned} \mathbb{E}^i : \Phi_l M^i &\equiv_{\prec(p^i, q^i)} M^i \Phi_{\underline{n}}, & \tilde{\mathbb{E}}^i : \Phi_l \tilde{M}^i &\equiv_{\prec(p^i, q^i)} \tilde{M}^i \Phi_{\tilde{n}}; \\ \mathbb{E}_\tau^i : 0 &\equiv_{\prec(p^i, q^i)} M^i \Phi_{\underline{n}}, & \tilde{\mathbb{E}}_\tau^i : 0 &\equiv_{\prec(p^i, q^i)} \tilde{M}^i \Phi_{\tilde{n}}. \end{aligned} \quad (5.3-5)$$

Remark 5.3.1 Remark 5.1.4 (i)-(iv) are still valid if we consider the matrices M^i and \tilde{M}^i over $k[x, \tilde{\phi}^i(x)^{-1}]$ instead of over k .

Theorem 5.3.2 There exists a unique reduction sequence based on the sequence $(*)'$:

$$\mathfrak{A} = \tilde{\mathfrak{A}}^0, \tilde{\mathfrak{A}}^1, \dots, \tilde{\mathfrak{A}}^s, \tilde{\mathfrak{A}}^{s+1}, \dots, \tilde{\mathfrak{A}}^\epsilon, \dots, \tilde{\mathfrak{A}}^t = \tilde{\mathfrak{A}}' \quad (*')$$

where the first part of the sequence till to $\tilde{\mathfrak{A}}^s$ is given by Theorem 5.2.1; the reduction from $\tilde{\mathfrak{A}}^s$ to $\tilde{\mathfrak{A}}^{s+1}$ is given by a loop mutation $a_1^{s+1} \mapsto (x)$, or first an edge reduction (0), then a loop mutation (x) ; the reduction from $\tilde{\mathfrak{A}}^i$ to $\tilde{\mathfrak{A}}^{i+1}$ for $s+1 \leq i < t$ is given by a regularization, or two regularization, or an edge reduction (0) then a regularization.

Proof We make reduction from $\tilde{\mathfrak{A}}^s$ to $\tilde{\mathfrak{A}}^{s+1}$ by a loop mutation when x does not locate at the \mathfrak{q}_Z -the block column of the formal product Θ^{s+1} , or by an edge reduction (0) then a loop mutation when x locates at the \mathfrak{q}_Z -column by Corollary 2.4.4.

Now suppose we have an induced bi-module problem $\tilde{\mathfrak{A}}^i$ for some $i > s$. If the first arrow a_1^i of \mathfrak{A}^i does not locate at the $(q+1)$ -th column of Θ^i , make a regularization $\tilde{a}_1^i \mapsto \emptyset$. Otherwise, there are two possibilities: ① there exists some $1 \leq j \leq n_Z$, the $(p^i, q+j)$ -th equation is a linear combination of the previous equations in \mathbb{E}_τ^i , then $\delta(\tilde{a}_0^i) = 0$ by Remark 5.3.1 and Corollary 2.4.4, set $\tilde{a}_0^i \mapsto (0)$, $\tilde{a}_1^i \mapsto \emptyset$; ② otherwise $\delta(\tilde{a}_0^i) \neq 0$, set $\tilde{a}_0^i \mapsto \emptyset$ and $\tilde{a}_1^i \mapsto \emptyset$. The sequence $(*)'$ is completed by induction as desired.

Corollary 5.3.3 \mathfrak{B}^t at $(*)'$ satisfying MW5 implies $\delta(\tilde{a}_0^\epsilon) = 0$ in $\tilde{\mathfrak{B}}^\epsilon$ at $(\tilde{*})'$.

Proof The first arrow a_1^t of \mathfrak{B}^t locates at the $(p, q+j)$ -th position, therefore $\delta^0(a_j^\epsilon) = 0$ in Θ^ϵ . By Remark 5.3.1, the (p, l) -th equation is a linear combination of previous equations in $\tilde{\mathbb{E}}_\tau^\epsilon$ for $0 \leq l \leq n_Z$. Thus $\delta(\tilde{a}_0^\epsilon) = 0$ in $\tilde{\mathfrak{B}}^\epsilon$ by Corollary 2.4.4, the proof is finished.

Proposition 5.3.4 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ with R trivial be a bipartite matrix bi-module problem having RDCC condition. If there exists an induced pair $(\mathfrak{A}', \mathfrak{B}')$ of $(\mathfrak{A}, \mathfrak{B})$ in the case of MW5, and $H' + \Theta'$ satisfies Classification 5.1.1 (I), then \mathfrak{A} is not homogeneous.

Proof Suppose we have a sequence $(*)'$ with $\mathfrak{B}' = \mathfrak{B}^t$, then there is a sequence $(\tilde{*})'$ based on $(*)'$ by Theorem 5.3.2. Corollary 5.3.3 tells that the first arrow \tilde{a}_0^ϵ of $\tilde{\mathfrak{B}}^\epsilon$ is an edge with $\delta(\tilde{a}_0^\epsilon) = 0$, and hence we may set $\tilde{a}_0^\epsilon \mapsto (1)$ according to Proposition 2.2.7. Denote by $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{B}})$ the induced pair, which is obviously local. Thus we are able to use the triangular formulae given in the section 3.3, and obtain an induced pair in one case of Classification 3.3.3 (ii)-(iv).

Case 1. If we meet 3.3.3 (ii), then $\tilde{\mathfrak{B}}^\epsilon$ is in the case of Proposition 3.4.5. We are done.

Case 2. If we meet 3.3.3 (iii), then there exists an induced boc of $\tilde{\mathfrak{B}}$ satisfying MW3, we are done by Proposition 3.4.3.

Case 3. If we meet 3.3.3 (iv), then there exists an induced bocs satisfying MW4, we are done by Proposition 3.4.4.

Case 4. If we meet of 3.3.3 (iv), and any induced minimal wild bocs satisfying MW5, then we choose one of them, say $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$, where the first arrow \hat{a}_1 in the sense of MW5 locates at the p^1 -th row in the formal product $\hat{\Theta}^1$. We claim that $p^1 < p$. In fact, the arrows \tilde{a}_j for $j = 1, \dots, n_z$ at the p -th row in $\hat{\Theta}$ of the pair $(\hat{\mathfrak{A}}, \hat{\mathfrak{B}})$ has the differentials $\delta^0(\tilde{a}_j) = s_{0j} + \dots$, and hence will be regularized according to Corollary 5.2.2.

Repeating the above process for $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$, if we meet one of the cases 1-3, the procedure stops. Otherwise if we meet the case 4 repeatedly, there is a sequence of local pairs and a decreasing sequence of the row indices:

$$\begin{array}{ccccccc} (\hat{\mathfrak{A}}, \hat{\mathfrak{B}}), & (\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1), & (\hat{\mathfrak{A}}^2, \hat{\mathfrak{B}}^2), & \dots, & (\hat{\mathfrak{A}}^\beta, \hat{\mathfrak{B}}^\beta), \\ p > & p^1 > & p^2 > & \dots > & p^\beta. \end{array}$$

Since the number of the rows of \hat{H}^i for $i = 1, \dots, v$ is fixed, the procedure must stop at some stage β , such that one of the Cases 1-3 appears. The proposition is proved by induction.

5.4 Bordered matrices in one-sided case

This subsection is devoted to constructing a reduction sequence starting from a one sided pair based on some bordered matrices.

Let $(\mathfrak{A}, \mathfrak{B})$ be a bipartite pair having RDCC condition, let $(\mathfrak{A}^r, \mathfrak{B}^r)$ be an induced matrix bi-module problem with R^r trivial given by Formula (4.1-1). Which gives a quotient-sub pair $((\mathfrak{A}^r)^{[m]}, (\mathfrak{B}^r)^{(m)})$ denoted by $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$. $\bar{\mathfrak{B}}$ has a layer $L = (R; \omega; d_1, \dots, d_m; v_1, \dots, v_t)$ by Definition 4.1.2. Denote by $\bar{T}_R = \{0\}$ and $\bar{T}_C = \{1, 2, \dots, m\}$, the row and column indices of (d_1, d_2, \dots, d_m) in the reduced formal product $\bar{\Theta}$ in Formula (4.1-2), then the vertex set $\bar{T} = \bar{T}_R \times \bar{T}_C$.

Let $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$ be an induced pair of $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$, with the induced functor $\bar{\vartheta} : R(\bar{\mathfrak{A}}') \rightarrow R(\bar{\mathfrak{A}})$. Write $\bar{M} = \bar{\vartheta}(F(k)) = \sum_{j=1}^m \bar{M}_j * E_j \in R(\bar{\mathfrak{A}})$ with the size vector $\underline{n} = (n_0; n_1, \dots, n_m)$ over \bar{T} .

Remark 5.4.1 Based on Formula (2.4-5) and using the reduced form parallel to Formula (4.1-7), we will construct the defining system $\bar{\mathbb{F}}$ of the quotient-sub-pair $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$.

(i) According to Remark 4.1.1 (i), denote by \bar{Z}_0 the (p^r, p^r) -the square block of $\Psi_{\bar{m}^r}$ of size n_0 with $p^r \in X^r$; by $\bar{Z}_{\xi\xi}$ the $(q^r + \xi, q^r + \xi)$ -th square block of size n_ξ with $q^r + \xi \in Y_i^r$, then

$$\bar{Z}_0 = Z_{X^r} = (z_{pq}^{X^r})_{n_0 \times n_0}, \quad \bar{Z}_{\xi\xi} = Z_{Y_i^r} = (z_{pq}^{Y_i^r})_{n_\xi \times n_\xi}.$$

(ii) Denote by \bar{Z}_ξ the $(p^r, q^r + \xi)$ -block of $\Psi_{\bar{m}^r}$ of size $n_0 \times n_\xi$, and by $\bar{Z}_{\eta\xi}$ the $(q^r + \eta, q^r + \xi)$ -block of size $n_\eta \times n_\xi$, $\eta < \xi$. Write the variable matrix $Z_j = (z_{pq}^j)$ for $j = 1, \dots, t^r$, then

$$\begin{aligned} \bar{Z}_\xi &= \sum_j \alpha_\xi^j Z_j, & \text{where } j \text{ runs over } s(V_j^r) \ni p^r, e(V_j^r) \ni q^r + \xi, & \alpha_\xi^j \in k; \\ \bar{Z}_{\eta\xi} &= \sum_j \beta_{\eta\xi}^j Z_j, & \text{where } j \text{ runs over } s(V_j^r) \ni q^r + \eta, e(V_j^r) \ni q^r + \xi, & \beta_{\eta\xi}^j \in k. \end{aligned}$$

Fix an integer $l \in \{1, \dots, m\}$ with $l \in Y \neq X$ in Definition 4.1.2, thus $d_l : X \rightarrow Y$ is a solid edge. Let $\rho \in Y$, such that $(p_\rho, q_\rho + 1)$ is the leading position of $\bar{H}'(k)$ in $\bar{\mathfrak{A}}'$, and suppose that $(q_\rho + 1)$ is the index of the first column of the ρ -th block-column over \bar{T} . Write

$$\mathbb{F} : (\bar{Z}_0 * E_0) \bar{M} \equiv_{\prec(p_\rho, q_\rho + 1)} \sum_{\xi=1}^m \bar{Z}_\xi * E_\xi + \bar{M} (\sum_{1 \leq \eta \leq \xi \leq m} \bar{Z}_{\eta\xi} * E_{\eta\xi}). \quad (5.4-1)$$

Similar as in Equation (5.1-3), we have the right hand side of $\bar{\mathbb{F}}$:

$$\mathbb{F}_\tau : 0 \equiv_{\prec(p_\rho, q_\rho + 1)} \sum_{\xi=1}^m \bar{Z}_\xi * E_\xi + \bar{M} (\sum_{1 \leq \eta \leq \xi \leq m} \bar{Z}_{\eta\xi} * E_{\eta\xi}). \quad (5.4-2)$$

Define a size vector $\tilde{n} = (\tilde{n}_0; \tilde{n}_1, \dots, \tilde{n}_m)$ over $\bar{\mathcal{T}}$ as follows: $\tilde{n}_\xi = n_\xi$ if $\xi \notin Y$; $\tilde{n}_\xi = n_\xi + 1$ if $\xi \in Y$. Construct a representation based on \tilde{M} :

$$\tilde{M} = \sum_{j=1}^m \tilde{M}_j * E_j \in R(\bar{\mathfrak{A}}), \quad \tilde{M}_j = \begin{cases} \tilde{M}_j, & \text{if } E_j 1_Y = 0; \\ (0 \ \tilde{M}_j), & \text{if } E_j 1_Y = E_j, \end{cases} \quad (5.4-3)$$

with 0 a column vector. Write $\tilde{Z}_0, \tilde{Z}_{\xi\xi}$ the variable matrices of size $\tilde{n}_0 \times \tilde{n}_0, \tilde{n}_\xi \times \tilde{n}_\xi$; and $\tilde{Z}_\xi = \sum_j \alpha_\xi^j \tilde{Z}_j$ of size $\tilde{n}_0 \times \tilde{n}_\xi$, $\tilde{Z}_{\eta\xi} = \sum_{\eta\xi}^j \beta_{\eta\xi}^j \tilde{Z}_j$ of size $\tilde{n}_\eta \times \tilde{n}_\xi$ according to Remark 5.4.1 respectively. Thus we obtain the following matrix equation with \tilde{q}_ρ being the index of the first column of the ρ -th block-column of \tilde{M} :

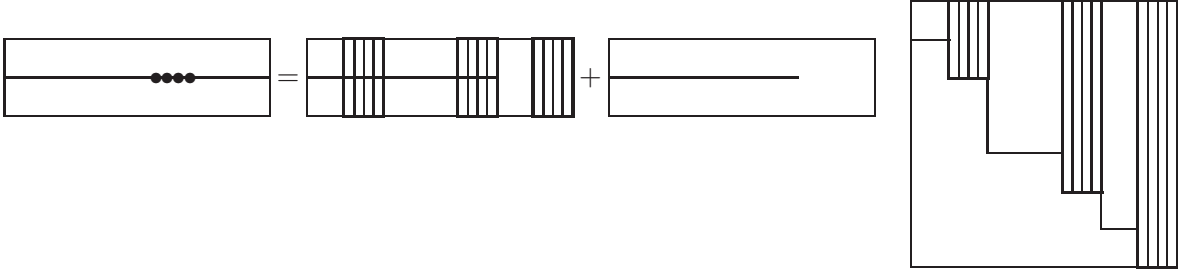
$$\begin{aligned} \tilde{\mathbb{F}} : \quad (\tilde{Z}_0 * E_0) \tilde{M} &\equiv_{\prec(p_\rho, \tilde{q}_\rho)} \sum_{\xi=1}^m \tilde{Z}_\xi * E_\xi + \tilde{M} (\sum_{1 \leq \eta \leq \xi \leq m} \tilde{Z}_{\eta\xi} * E_{\eta\xi}), \\ \tilde{\mathbb{F}}_\tau : \quad 0 &\equiv_{\prec(p_\rho, \tilde{q}_\rho)} \sum_{\xi=1}^m \tilde{Z}_\xi * E_\xi + \tilde{M} (\sum_{1 \leq \eta \leq \xi \leq m} \tilde{Z}_{\eta\xi} * E_{\eta\xi}). \end{aligned} \quad (5.4-4)$$

Taken any integer $p' \geq p_\rho$ and $1 \leq h \leq n_\rho$, the $(p', q_\rho + h)$ -th entry of \mathbb{F}_τ equals

$$\sum_{p'} \gamma_{p,p'} z_{p',q_\rho+h}^{Y^r} + \sum_{p'} \nu_{p,p'} z_{p',q_\rho+h}^j, \quad \gamma_{p,p'}, \nu_{p,p'} \in k. \quad (5.4-5)$$

It is clear that $z_{p',q_\rho+h_1}^Y$ and $z_{p',q_\rho+h_2}^Y$, $z_{p',q_\rho+h_1}^j$ and $z_{p',q_\rho+h_2}^j$ in Formula (5.4-5) have the same coefficients respectively for all $1 \leq h_1, h_2 \leq n_\rho, h_1 \neq h_2$. The assertion is also valid for $\tilde{\mathbb{F}}_\tau$.

The picture below shows four equations (abridged by four circles) of \mathbb{F}_τ in Formula (5.4-2). There are three solid edges ending at Y , where $|Y| = 3$, $n_\rho = 4$, the equations locate at the $(p_2, q_2 + h)$ -th positions for $h = 1, 2, 3, 4$ have the same coefficients as shown in Formula (5.4-5).



Remark 5.4.2 Parallel to Remark 5.1.4, we have the following facts.

(i) For any $1 \leq h_1, h_2 \leq n_\rho$, the $(p_\rho, q_\rho + h_1)$ -th equation is a linear combination of the previous equations in \mathbb{F}_τ , if and only if so is the $(p_\rho, q_\rho + h_2)$ -th equation by Formula (5.4-5). Similarly, we have the same result in $\tilde{\mathbb{F}}_\tau$.

(ii) The equations in the system $\tilde{\mathbb{F}}$ (resp. $\tilde{\mathbb{F}}_\tau$) and those in \mathbb{F} (resp. \mathbb{F}_τ) are the same everywhere except at the \tilde{q}_ρ -column, since the entries of \tilde{M} at the \tilde{q}_ρ -th column is zero for $\rho = 1, \dots, |Y|$ respectively.

(iii) If there exists some $1 \leq h \leq n_\rho$, such that the $(p_\rho, q_\rho + h)$ -th equation is a linear combination of the previous equations in \mathbb{F}_τ , then so is the (p_ρ, \tilde{q}_ρ) -th equation in $\tilde{\mathbb{F}}$, by (ii) and (i) above.

(iv) If the $(p_\rho, q_\rho + h)$ -th equation is a linear combination of the previous equations in \mathbb{F} , then so is in \mathbb{F}_τ , since the variables in $\{\tilde{Z}_0, \tilde{Z}_{\rho\rho}, Z_j\}_j$ are algebraically independent.

Suppose there is a reduction sequence given by Formula (4.1-5) with $\bar{\mathfrak{A}}^0 = \bar{\mathfrak{A}}$ and each reduction being in the sense of Lemma 2.3.2:

$$\begin{aligned} \mathfrak{A}, \mathfrak{A}^1, \dots, \mathfrak{A}^{r-1}, \quad \mathfrak{A}^r, \quad \mathfrak{A}^{r+1}, \quad \dots, \quad \mathfrak{A}^{r+i}, \quad \mathfrak{A}^{r+i+1}, \quad \dots, \quad \mathfrak{A}^{r+s}; \\ \bar{\mathfrak{A}}^0, \quad \bar{\mathfrak{A}}^1, \quad \dots, \quad \bar{\mathfrak{A}}^i, \quad \bar{\mathfrak{A}}^{i+1}, \quad \dots, \quad \bar{\mathfrak{A}}^s, \end{aligned} \quad (*)$$

Proposition 5.4.3 Parallel to Theorem 5.2.1, there exists a unique reduction sequence based on the sequence $(\bar{*})$:

$$\begin{array}{cccccccc} \tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}^1, \dots, \tilde{\mathfrak{A}}^{r-1}, & \tilde{\mathfrak{A}}^r, & \tilde{\mathfrak{A}}^{r+1}, & \dots, & \tilde{\mathfrak{A}}^{r+i}, & \tilde{\mathfrak{A}}^{r+i+1}, & \dots, & \tilde{\mathfrak{A}}^{r+s}; \\ & \tilde{\mathfrak{A}}, & \tilde{\mathfrak{A}}^1, & \dots, & \tilde{\mathfrak{A}}^i, & \tilde{\mathfrak{A}}^{i+1}, & \dots, & \tilde{\mathfrak{A}}^s. \end{array} \quad (\tilde{*})$$

- (i) $\tilde{\mathfrak{A}}^i = \mathfrak{A}^i$ for $i = 0, 1, \dots, r$.
- (ii) The reduction from $\tilde{\mathfrak{A}}^i$ to $\tilde{\mathfrak{A}}^{i+1}$ is a reduction or a composition of two reductions in the sense of Lemma 2.3.2 for $i = 0, \dots, s-1$, such that $\tilde{\mathcal{T}}^s$ has two vertices, and $\tilde{\vartheta}^{0s}(\tilde{F}^s) = \tilde{M}^s$.
- (iii) The reduction from $\tilde{\mathfrak{A}}^{r+i}$ to $\tilde{\mathfrak{A}}^{r+i+1}$ is as the same as that from $\tilde{\mathfrak{A}}^i$ to $\tilde{\mathfrak{A}}^{i+1}$.
- (iv) The diagonal block \tilde{e}_X in $\tilde{\mathcal{K}}_0^s$ of $\tilde{\mathfrak{A}}^s$ over $\tilde{\mathcal{T}}$ is of the form given in Corollary 5.2.2.

Proof (i) is clear. The proof of (ii) is parallel to that of Theorem 5.2.1, the only difference is that for each $\rho \in Y$, we need to add a column into the ρ -th block column for each $\rho = 1, \dots, |\tilde{Y}|$ step by step. (iii) follows from Formula (4.1-5). The proof of (iv) is parallel to Corollary 5.2.2. The proof of the theorem is completed.

Parallel to $(*)'$ of the subsection 5.3, suppose we have the following sequences:

$$\begin{array}{cccccccc} \mathfrak{A}, \mathfrak{A}^1, \dots, \mathfrak{A}^{r-1}, & \mathfrak{A}^r, & \mathfrak{A}^{r+1}, & \dots, & \mathfrak{A}^{r+s}, & \mathfrak{A}^{r+s+1}, & \dots, & \mathfrak{A}^{r+\epsilon}, & \dots, & \mathfrak{A}^{r+t}; \\ & \mathfrak{A}, & \mathfrak{A}^1, & \dots, & \mathfrak{A}^s, & \mathfrak{A}^{s+1}, & \dots, & \mathfrak{A}^\epsilon, & \dots, & \mathfrak{A}^t, \end{array} \quad (*')$$

the reduction from $\tilde{\mathfrak{A}}$ (resp. \mathfrak{A}^r) to $\tilde{\mathfrak{A}}^s$ (resp. \mathfrak{A}^{r+s}) is given by $(\bar{*})$; from $\tilde{\mathfrak{A}}^s$ (resp. \mathfrak{A}^{r+s}) to $\tilde{\mathfrak{A}}^{s+1}$ (resp. \mathfrak{A}^{r+s+1}) is a loop mutation and we obtain a parameter x ; the reduction from $\tilde{\mathfrak{A}}^i$ (resp. \mathfrak{A}^{r+i}) to $\tilde{\mathfrak{A}}^{i+1}$ (resp. \mathfrak{A}^{r+i+1}) is a regularization for $i = s+1, \dots, t-1$. The pair $(\mathfrak{A}^{r+t}, \mathfrak{B}^{r+t})$ is minimally wild in the case of MW5 and Classification 5.1.1 (II).

Remark 5.4.4 (i) If the first arrow a_1^t of $\tilde{\mathfrak{B}}^t$ splits from d_l of the one sided sub-bocs $\tilde{\mathfrak{B}}$, then $d_l : X \mapsto Y$ is an edge by Theorem 4.6.1 and Corollary 4.6.2. Consequently we are able to add some columns according to Theorem 5.4.3.

(ii) We will discuss how to determine \mathfrak{A}^r in $(*)'$ in the next subsection.

(iii) Suppose a_1^t locates at the (p, q') -th position in the reduced formal product $\tilde{\Theta}^t$ with $q' = q + j$ for some $1 \leq j \leq n_l$, and the first arrow a_1^ϵ of $\tilde{\mathfrak{B}}^\epsilon$ locates at the $(p, q+1)$ -th position in $\tilde{\Theta}^\epsilon$. Let integers $\rho \in Y$, parallel to Formula (5.3-1) we write an equation system $\tilde{\mathbb{F}}_\tau^{(\bullet, \rho)}$. Thus we obtain a polynomial $d^{\eta\rho}(x)$ inductively for $\rho = 1, \dots, |Y|$, $\eta = p_x, \dots, p$, according to the discussion given in the first part of the subsection 5.3. If $\bar{R}^t = k[x, \phi^t(x)^{-1}]$, define

$$\tilde{\phi}^t(x) = \phi^t(x) \prod_{\eta=p_x}^p \prod_{\rho=1}^{|Y|} d^{\eta\rho}(x),$$

Proposition 5.4.5 Parallel to Theorem 5.3.4, there exists a unique reduction sequence based on the sequence $(\bar{*}')$ satisfying the proposition (i)-(iv) below:

$$\begin{array}{cccccccc} \tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}^1, \dots, \tilde{\mathfrak{A}}^{r-1}, & \tilde{\mathfrak{A}}^r, & \tilde{\mathfrak{A}}^{r+1}, & \dots, & \tilde{\mathfrak{A}}^{r+s}, & \tilde{\mathfrak{A}}^{r+s+1}, & \dots, & \tilde{\mathfrak{A}}^{r+\epsilon}, & \dots, & \tilde{\mathfrak{A}}^{r+t}; \\ & \tilde{\mathfrak{A}}, & \tilde{\mathfrak{A}}^1, & \dots, & \tilde{\mathfrak{A}}^s, & \tilde{\mathfrak{A}}^{s+1}, & \dots, & \tilde{\mathfrak{A}}^\epsilon, & \dots, & \tilde{\mathfrak{A}}^t. \end{array} \quad (\tilde{*}')$$

- (i) The first parts of the two sequences up to $r+s$ and s respectively are given by $(\tilde{*})$.
- (ii) $\tilde{\mathfrak{A}}^{s+1}$ is induced from $\tilde{\mathfrak{A}}^s$ by a loop mutation $a_1^{s+1} \mapsto (x)$, or an edge reduction (0), then a loop mutation (x) ; the reduction from $\tilde{\mathfrak{A}}^{s+i}$ to $\tilde{\mathfrak{A}}^{s+i+1}$ is given by a regularization, or a composition of two regularizations, or an edge reduction (0) then a regularization for $i = 1, \dots, t-1$.
- (iii) The reduction from $\tilde{\mathfrak{A}}^{r+s+i}$ to $\tilde{\mathfrak{A}}^{r+s+i+1}$ is as the same as that from $\tilde{\mathfrak{A}}^{s+i}$ to $\tilde{\mathfrak{A}}^{s+i+1}$ for $i = 1, \dots, (t-s-1)$.

(iv) Denote by $a_1^t \in \bar{\mathfrak{B}}^t$ the first solid arrow, if $\delta(a_1^t) = vx - xv$, then the first solid edge \tilde{a}_0^ϵ of $\tilde{\mathfrak{B}}^\epsilon$ with the differential $\delta(\tilde{a}_0^\epsilon) = 0$.

Proof (i) is obvious. The proof of (ii) is parallel to that of Theorem 5.3.2. (iii) follows from Formula (4.1-5). (iv) is parallel to Corollary 5.3.3. The proof is finished.

5.5 Non-homogeneity in the case of MW5 and classification (II)

Suppose a bipartite pair $(\mathfrak{A}, \mathfrak{B})$ has an induced pair $(\mathfrak{A}', \mathfrak{B}')$ in the case of MW5 and Classification 5.1.1 (II). This subsection is devoted to determining the one sided quotient-sub pair according to the position of the first arrow a_1' in the formal product $H' + \Theta'$, and then proving that $(\mathfrak{A}, \mathfrak{B})$ is not homogeneous.

Let \mathfrak{A} be a bipartite matrix bi-module problem having RDCC condition. The sequence

$$(\mathfrak{A}, \mathfrak{B}), (\mathfrak{A}^1, \mathfrak{B}^1), \dots, (\mathfrak{A}^\varsigma, \mathfrak{A}^\varsigma), (\mathfrak{A}^{\varsigma+1}, \mathfrak{B}^{\varsigma+1}), \dots, (\mathfrak{A}^\tau, \mathfrak{B}^\tau) = (\mathfrak{A}', \mathfrak{B}') \quad (5.5-1)$$

satisfies the following condition: R^i is trivial for $i \leq \varsigma$, the reduction from \mathfrak{A}^i to \mathfrak{A}^{i+1} is in the sense of Lemma 2.3.2 for $i < \varsigma$; \mathfrak{A}^ς is local with $\delta(a_1^\varsigma) = 0$, after a loop mutation, we have $R^{\varsigma+1} = k[x]$ in $\mathfrak{B}^{\varsigma+1}$; finally we make regularization for $i > \varsigma$, and $\mathfrak{B}^\tau = \mathfrak{B}'$ is in the case of MW5 and Classification 5.1.1 (II). Suppose the first arrow a_1^τ of \mathfrak{B}^τ locates at the (p^τ, q^τ) -th position of the (\mathbf{p}, \mathbf{q}) -th block in the formal product Θ^τ . Since the size of H^ς coincides with that of H^τ , and since we make regularization from $\mathfrak{B}^{\varsigma+1}$ to \mathfrak{B}^τ , according to Formula (2.3-5):

$$H^\tau(k[x, \phi^\tau(x)^{-1}]) = \sum_{i=1}^\varsigma G_\tau^i * A_1^{i-1} + (x) * A_1^\varsigma. \quad (5.5-2)$$

From now on, we call G_τ^i , the leading block of $G_\tau^i * A_1^{i-1}$, a *G-type matrix* of $H^\tau(k[x, \phi^\tau(x)^{-1}])$, and sometimes do not distinguish G^i over R^i or $G^i(k)$ over k .

Let $i < \varsigma$, $\vartheta^{i\tau} : R(\mathfrak{A}^\tau) \rightarrow R(\mathfrak{A}^i)$ be the induced functor, and $\underline{n}^{i\tau} = \vartheta^{i\tau}(1, 1, \dots, 1)$. There is a simple fact, that any row (column) index ρ of $H^i + \Theta^i$ in the pair $(\mathfrak{A}^i, \mathfrak{B}^i)$ determines a row (column) index $n_1^{i\tau} + \dots + n_\rho^{i\tau}$ of $H^\tau + \Theta^\tau$ in the pair $(\mathfrak{A}^\tau, \mathfrak{B}^\tau)$. Consequently, if the upper (resp. lower, left or right) boundaries of two *G-type matrices* $G^i(1), G^i(2)$ of $H^i(k)$ are colinear, then the same boundaries of two splitting blocks $G_\tau^i(1), G_\tau^i(2)$ of $H^\tau(k[x, \phi(x)^{-1}])$ are still colinear.

Remark 5.5.1 Consider the *G-type matrices* inside the (\mathbf{p}, \mathbf{q}) -th block. If G_τ^i and G_τ^{i+1} are both in the (\mathbf{p}, \mathbf{q}) -th block, the relative position of their upper boundaries has three possibilities according to Formulae (2.3-1)-(2.3-3).

(i) The upper boundaries of G_τ^i and G_τ^{i+1} are co-linear, if and only if the reduction from \mathfrak{A}^{i-1} to \mathfrak{A}^i is given by one of the following: $G^i = (0), (1), (0 \ 1)$; $G^i = (\lambda)$; $G^i = \emptyset$; and the right boundary of G_τ^i is not that of the (\mathbf{p}, \mathbf{q}) -th block. In this case their lower boundaries are also colinear.

(ii) The upper boundary of G_τ^{i+1} is strictly lower than that of G_τ^i , if and only if $G^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \ 1 \\ 0 \end{pmatrix}$; or $G^i = W$ of size being strictly bigger than 1, and the right boundary of G_τ^i is not that of the (\mathbf{p}, \mathbf{q}) -th block. In this case, the lower boundaries of G_τ^i and G_τ^{i+1} are also colinear.

(iii) The lower boundary of G_τ^{i+1} is the upper boundary of G_τ^i , if and only if the right boundary of G_τ^i coincides with that of the (\mathbf{p}, \mathbf{q}) -th block.

Collect all the *G-type matrices* of H^τ inside the (\mathbf{p}, \mathbf{q}) -th block, such that their upper boundaries are above or at that of a_1^τ :

$$G_\tau^{q_1}, G_\tau^{q_2}, \dots, G_\tau^{q_u}, \quad \text{with} \quad q_1 < q_2 < \dots < q_u < \varsigma. \quad (5.5-3)$$

The G -type matrices $G_\tau^{q_i}$ ($1 \leq i \leq u$) in (5.5-3) are grouped into h groups according to their upper boundaries are colinear or not, and denoted by ρ_j the common upper boundary of the j -th group for $j = 1, \dots, h$, where ρ_{j+1} is strictly lower than ρ_j :

$$\{G_\tau^{q_{11}}, \dots, G_\tau^{q_{1,u_1}}\}, \dots, \{G_\tau^{q_{h,1}}, \dots, G_\tau^{q_{h,u_h}}\}, \quad u_1 + \dots + u_h = u. \quad (5.5-4)$$

The matrices $G_\tau^{q_{j,l}}$ and $G_\tau^{q_{j,l+1}}$ in the j -th group have two possibilities: ① If $G_\tau^{q_{j,l}}$ is in the case of Remark 5.5.1 (i), then $G_\tau^{q_{j,l+1}}$ comes from the next reduction with $q_{j,l+1} = q_{j,l} + 1$. ② If $G_\tau^{q_{j,l}}$ is in the case of Remark 5.5.1 (ii), then $G_\tau^{q_{j,l+1}}$ follows by a sequence of reductions with the upper boundary of the G -type matrices lower than that of the p^τ -row, and including at least one reduction in the case of Remark 5.5.1 (iii). At last the sequence reaches $G_\tau^{q_{j,l+1}}$ with the upper boundary ρ_j as a neighbor of $G_\tau^{q_{j,l}}$, thus $q_{j,l+1} > q_{j,l} + 1$.

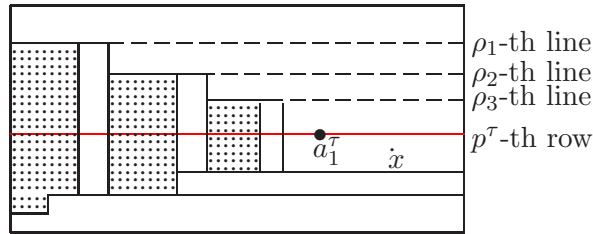
Lemma 5.5.2 $G_\tau^{q_{j,u_j}}$ must be in the case of Remark 5.5.1 (ii) for $j = 1, \dots, h$.

Proof If $G_\tau^{q_{j,u_j}}$ is in the case of 5.5.1 (iii), then ρ_j is lower than ρ_{j+1} , a contradiction to the grouping of Formula (5.5-4); and a_1^τ is sitting upper ρ_h for $j = h$, a contradiction to the choice of the sequence (5.5-3).

Suppose $G_\tau^{q_{j,u_j}}$ satisfies 5.5.1 (i). Then for $j < h$, the upper boundaries of $G_\tau^{q_{j,u_j}}$ and $G_\tau^{q_{j,u_j}+1}$ coincide, a contradiction to the grouping in (5.5-4). For $j = h$, $G_\tau^{q_{h,u_h}} = 0, I, (0I)$, or λI , or \emptyset with hight $d \geq 1$. Suppose the next reduction gives $G_\tau^{q_{h,u_h}+1}$ and denoted by G'_τ for simplicity. If G'_τ satisfies 5.5.1 (i) and (ii), then G'_τ and $G_\tau^{q_{h,u_h}}$ have the same upper boundary, a contradiction to the grouping of (5.5-4); if G'_τ satisfies 5.5.1 (iii), then a_1^τ locates above ρ_h , a contradiction to the choice of (5.5-3). Therefore there is no any further reduction in the sense of Lemma 2.3.2. If the hight $d > 1$, \mathfrak{B}^τ is not local, so $d = 1$. Since a_1^τ locates between the lower and the upper boundary of $G_\tau^{q_{h,u_h}}$, which forces $G_\tau^{q_{h,u_h}}$ sitting at the p^τ -th row. But the parameter x appears after $G_\tau^{q_{h,u_h}}$ and before a_1^τ , thus locates at the p^τ -th row, a contradiction to Lemma 5.1.2. Therefore $G_\tau^{q_{j,u_j}}$ is in the case of 5.5.1 (ii), the proof is completed.

Definition 5.5.3 We define h rectangles in Θ^τ : for $j < h$, the j -th rectangle has the upper boundary ρ_j , lower boundary ρ_{j+1} , and the left boundary is the right boundary of $G_\tau^{q_{j,u_j}}$, the right boundary is that of the (p, q) -th block. While the h -th rectangle has the upper boundary ρ_h , lower boundary is that of $G_\tau^{q_{h,u_h}}$. The rectangle with the upper boundary ρ_j is said to be the j -th *lader*, there are altogether h laders.

The picture below shows an example, in which $h = 3$, the three groups given by sawtooth patterns with some dots, and the last G -type matrix in each group is given by a rectangle without dots. The upper boundaries of the three laders are shown by dotted lines.



Lemma 5.5.4 Let $r = q_{h,u_h} - 1$ in the formula (5.5-1). We define a one sided quotient-sub pair $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{B}}) = ((\mathfrak{A}^r)^{[m]}, (\mathfrak{B}^r)^{(m)})$ of the pair $(\mathfrak{A}^r, \mathfrak{B}^r)$ consisting of the solid arrows d_1, \dots, d_m sitting at the p^τ -row with the right boundary of d_m is that of the (p, q) -th block as shown in Formula (4.1-1). Then

- (i) $m > 1$;

- (ii) all the G -type matrices in H^τ coming from d_2, \dots, d_m locate below the p^τ -th row;
- (iii) a_1^τ is split from d_l with $l > 1$. If $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ satisfies Theorem 4.6.1 or Corollary 4.6.2, then d_l is a solid edge.
- (iv) $\varsigma = r + s$ and $\tau = r + t$ in the formula (5.5-1), which coincides with sequence $(\bar{*}')$ given above Remark 5.4.4.

Proof (i) follows from Lemma 5.5.2. (ii) comes from the choice of the sequence (5.5-3). (iii) and (iv) are obvious. The proof is finished.

Lemma 5.5.5 (i) Let $(\mathfrak{A}_X^r, \mathfrak{B}_X^r)$ be the induced local pair at X of $(\mathfrak{A}^r, \mathfrak{B}^r)$ defined above, denote by h_X the number of the inheriting ladders in $H_X^r + \Theta_X^r$, then $h_X \leq h$.

(ii) $H^r + \Theta^r$ of the pair $(\mathfrak{A}^r, \mathfrak{B}^r)$ has also h ladders in the (p, q) -th block. The boundaries of the j -th ladder of $H^r + \Theta^r$ comes from that of the j -th ladder of $H^r + \Theta^r$ for $j = 1, \dots, h$, according to the simple fact above Remark 5.5.1.

(iii) Return to the formulae $(\bar{*}')$ and $(\tilde{*}')$ of Proposition 5.4.5, then $\tilde{H}^{r+t} + \tilde{\Theta}^{r+t}$ has also h ladders. Furthermore, the number of rows in the h -th (resp. in the j -th, for $j = 1, \dots, h-1$) ladder in $\tilde{H}^{r+t} + \tilde{\Theta}^{r+t}$ is as the same as (resp. as the same as or more than) that in $H^{r+t} + \Theta^{r+t}$.

Proposition 5.5.6 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a bipartite matrix bi-module problem having RDCC condition. If there exists an induced pair $(\mathfrak{A}', \mathfrak{B}')$ of $(\mathfrak{A}, \mathfrak{B})$, which satisfies MW5 and $H' + \Theta'$ of $(\mathfrak{A}', \mathfrak{B}')$ is in the case of Classification 5.1.1 (II), then \mathfrak{A} is not homogeneous.

Proof Suppose the induced pair $(\mathfrak{A}', \mathfrak{B}')$ is the last term $(\mathfrak{A}^{r+t}, \mathfrak{B}^{r+t})$ of the sequence $(\bar{*}')$ above Remark 5.4.4. We assume in addition that the number of the ladders in $H^{r+t} + \Theta^{r+t}$ is minimal with the property of MW5 and Classification 5.1.1 (II).

(I) Let X be given by Definition 4.1.2. If $(\mathfrak{A}_X^r, \mathfrak{B}_X^r)$ is wild, denote by $((\mathfrak{A}_X^r)', (\mathfrak{B}_X^r)')$ the induced minimally wild local pair obtained by using the triangular Formulae of subsection 3.3 with the parameter x' and the first arrow a_1' .

(I-1) If $(\mathfrak{B}_X^r)'$ is of MW3, MW4, or MW5 with $H_X' + \Theta_X'$ being in the case of Classification 5.1.1 (I), then it is not homogeneous by Proposition 3.5.3-3.5.5, we are done.

(I-2) If $(\mathfrak{B}_X^r)'$ is in the case of MW5 and Classification 5.1.1 (II), then, the number of the inheriting ladders h_X in $H_X^r + \Theta_X^r$ with $h_X \leq h$ by Lemma 5.5.5 (i). Suppose a_1' locates at the h' -ladder. If $h' = h_X = h$, since this ladder contains only one row by Lemma 5.5.4, x' must locate at the same row, a contradiction to Lemma 5.1.2.; if $h' < h_X = h$, or $h' = h_X < h$, then it contradicts to the minimality assumption on the number of ladders.

(II) Suppose $(\mathfrak{A}_X^r, \mathfrak{B}_X^r)$ is tame infinite, the quotient-sub-pair $(\bar{\mathfrak{A}}_X, \bar{\mathfrak{B}}_X)$ is in the case of Classification 4.2.1 (ii).

(II-1) If the one sided pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ satisfies the hypothesis of Lemma 4.2.3 or 4.4.1, since the unique effective loop \bar{b} of $\bar{\mathfrak{B}}_X$ is as the same as that of \mathfrak{B}_X^r , \mathfrak{B}^r is not homogeneous.

(II-2) If $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ satisfies Theorem 4.4.2, then we use triangular formulae of the subsection 3.3 for the local wild pair $(\mathfrak{A}^{r+2l}, \mathfrak{B}^{r+2l})$. If we reach the cases of MW3, MW4, or MW5 and Classification 5.1.1 (I), we are done. If we meet again MW5 and Classification 5.1.1 (II), the first arrow must be outside of the h -th ladder, a contradiction to the minimality assumption.

(III) Now we consider the following two cases: (i) \mathfrak{B}_X^r is tame infinite, $\bar{\mathfrak{B}}_X$ satisfies Classification 4.2.1 (ii), the pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ is major and satisfies Formula (4.2-6); (ii) \mathfrak{B}_X^r is tame infinite or finite, and $\bar{\mathfrak{B}}_X$ is finite. Then in both cases d_l of $\bar{\mathfrak{B}}$, from which a_1^t split, is a solid edge by Lemma 5.5.4 (iii). Consequently, Formula $(\tilde{*})$ of Proposition 5.4.3 can be used with respect to d_l . Furthermore by Formula $(\tilde{*}')$ of Proposition 5.4.5, $\delta(\tilde{a}_0^\epsilon) = 0$ in $\tilde{\mathfrak{A}}^\epsilon$ by 5.4.5 (iv). Set the edge $\tilde{a}_0^\epsilon \mapsto (1)$ then all the other arrows split from d_l at the same row are mapped to \emptyset by Proposition 5.4.3 (iv). The induced pair is obviously local of tame infinite or wild type, then we are able to use the triangular formulae once again, and obtain an induced pair $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ in the cases (ii)-(iv) of Classification 3.3.3.

(III-1) If the induced local pair $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ is tame infinite, then Proposition 3.5.5 ensures that the two-point pair $(\tilde{\mathfrak{A}}^{r+\epsilon}, \tilde{\mathfrak{B}}^{r+\epsilon})$ is wild and non-homogeneous, we are done.

(III-2) If $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ is in the case of MW3, or MW4, or MW5 and Classification 5.1.1 (I), then it is non-homogeneous, we are done.

(III-3) If $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ is in the case of MW5 and classification 5.1.1 (II), and suppose in addition, whose first arrow locates at the h_1 -th ladder with $h_1 < h$, which contradicts to the minimality number assumption of the ladders.

(III-4) If $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ is in the case of MW5 and classification 5.1.1 (II), and suppose in addition, whose first arrow locates still at the h -th ladder. We need to do induction on some pairs of integers. Denote by σ the number of the rows in the h -th ladder of $H^{r+t} + \Theta^{r+t}$, which is a constant after making some bordered matrices by Lemma 5.5.5 (iii); and m , the number of the solid arrows in the pair $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{B}})$, is also a constant. Define a finite set with σm pairs:

$$\mathcal{S} = \{(\varrho, \zeta) \mid 1 \leq \varrho \leq \sigma, \zeta = 1, \dots, m\},$$

ordered by $(\varrho_1, \zeta_1) \prec (\varrho_2, \zeta_2) \iff \varrho_1 > \varrho_2, \text{ or } \varrho_1 = \varrho_2, \zeta_1 < \zeta_2.$

Denote the induced minimally wild local pair $(\mathfrak{A}^{r+t}, \mathfrak{B}^{r+t})$ in $(\bar{*}')$ by $(\hat{\mathfrak{A}}^0, \hat{\mathfrak{B}}^0)$ for unifying the notations. Let $(\varrho^0, \zeta^0) \in \mathcal{S}$, such that $\varrho^0 = p$ is the row-index of the first arrow $\hat{a}_1^0 = a_1'$ in the h -th ladder by Remark 5.4.4 (iii); $\zeta^0 = l$, since a_1' splits from the edge d_l by Lemma 5.5.4 (iii). Similarly denote by $(\varrho^1, \zeta^1) \in \mathcal{S}$ determined by the first arrow $\hat{a}_1^1 \in \hat{\mathfrak{B}}^1$. Proposition 5.4.3 (iv) ensures $(\varrho^0, \zeta^0) \prec (\varrho^1, \zeta^1)$.

Now we start the procedure (III) from the pair $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ instead of $(\hat{\mathfrak{A}}^0, \hat{\mathfrak{B}}^0)$. If (III-4) appears repeatedly, then after finitely many steps, we reach an induced pair of (III-1)-(III-3) by induction on \mathcal{S} . The proof is completed.

5.6 The proof of the main Theorem

We are ready to prove the main theorem 3.

Theorem 5.6.1 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a bipartite matrix bi-module problem having RDCC condition. If \mathfrak{A} is of wild type, then $R(\mathfrak{A})$ is not homogeneous.

Proof For any wild boc, there exists an induced \mathfrak{B}' satisfying one of MW1-MW5 according to Classification 3.3.2. Then Proposition 3.4.1-3.4.4 proved that \mathfrak{B}' in the case of MW1-MW4 is not homogeneous. When \mathfrak{B} is the associated boc of a bipartite matrix bi-module problem having RDCC condition, Proposition 5.3.4 and 5.5.6 proved that the induced pair $(\mathfrak{A}', \mathfrak{B}')$ in the case of MW5 is not homogeneous. Therefore $(\mathfrak{A}, \mathfrak{B})$ is not homogeneous, the proof is completed.

Proof of Main Theorem 3 Let Λ be a finite-dimensional basic algebra over an algebraically closed field k . If Λ is of wild representation type, then $\text{mod-}\Lambda$ is not homogeneous.

In fact, let \mathfrak{A} be the matrix bi-module problem associated to Λ . Then \mathfrak{A} is bipartite, having RDCC condition by Remark 1.4.4, and is representation wild type. So the pair $(\mathfrak{A}, \mathfrak{B})$ is not homogeneous by Theorem 5.6.1. Note that there is an almost one-to-one correspondence between almost split sequences in $\text{mod-}\Lambda$ and almost split conflations in $R(\mathfrak{B})$, see [B2] and [ZZ], therefore $\text{mod-}\Lambda$ is not homogeneous. The proof is finished.

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